Instructions: There are three parts. You must turn in 1 problem from Part I, 2 problems from Part II, and both problems from Part III. All problems have equal weight. Each problem submitted must be written on a separate sheet of paper with your name and problem number at the top. Your solutions must be complete and clear. You must show that the hypotheses of well known results that you use are satisfied. Make sure they are properly used and quoted. *Unless otherwise indicated* all references to measure and integration on the real line is in the sense of Lebesgue. We wish you well!

Part I: Choose 1 of the following 2 problems

1. Let μ be a *finite* and *finitely additive* measure on a field $\mathcal{A} \subset \mathcal{P}(X)$. (Recall that μ is finitely additive if $\mu(\cup_{i=1}^k A_i) = \sum_{i=1}^k \mu(A_i)$ for every finite disjoint collection of sets $\{A_i\}_{i=1}^k$.) Prove that μ is countably additive on $\mathcal{A}$ if and only if for each decreasing sequence
$$A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$$
with $\cap_{n=1}^\infty A_n = \emptyset$ we have $\lim_{n\to\infty}(A_n) = 0$.

2. (a) State Egoroff's theorem.
   (b) Give an example of a sequence of Lebesgue measurable functions $f_n : [0,1] \to \mathbb{R}$ that converges everywhere to a function f but there is no subset A of $[0,1]$ of Lebesgue measure 0 such that f_n converges uniformly to f on $[0,1] \setminus A$. Justify your example.

Part II: Choose 2 of the following 3 problems

3. Find
$$\lim_{k\to\infty} \int_0^k \frac{x^n}{n!}\left(1 - \frac{x}{k}\right)^k dx,$$
for all $n \geq 0$. You must completely justify your answer.

4. For each $n \geq 3$ let
$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x < \frac{1}{n} \\ 2n - n^2 x & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & \text{if } \frac{2}{n} < x \leq 1. \end{cases}$$
Suppose g is a continuous real-valued function on $[0,1]$. Show
$$\lim_{n\to\infty} \int_0^1 f_n(x) g(x)\, dx = g(0).$$

5. Let f_n, $n \geq 1$, be a sequence of integrable function on $\mathbb{R}$ such that $f_n(x) \geq f_{n+1}(x) \geq 0$, for each $x \in \mathbb{R}$ and each $n \geq 1$. If
$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n\, dx = 0,$$
prove that $\lim_{n\to\infty} f_n = 0$ almost everywhere on $\mathbb{R}$.

Part III: The following 2 problems are mandatory.

6. Suppose f is absolutely continuous on $[0, 1]$. Suppose there is a continuous function g on $[0, 1]$ so that $g(x) = f'(x)$ almost everywhere. Prove that f is differentiable at each point in $[0, 1]$ and $f'(x) = g(x)$, $0 \leq x \leq 1$.

7. Let $f \in L^2(\mathbb{R})$, (with respect to Lebesgue measure). Prove that f can be represented in the form
$$f = g + h,$$
where $g \in L^1(\mathbb{R})$ and $h \in L^\infty(\mathbb{R})$.