Comprehensive/Qualifying Examination

Real Analysis

August 2018

Instructions. You must solve 2 problems from **Part I**, 2 problems from **Part II**, and 1 problem from **Part III**. All problems have equal weight. Each solution submitted must be written on a separate sheet of paper with your name and problem number at the top. Indicate on a separate sheet the problems you **omit**.

Carefully show all your steps. You may appeal to a "well known theorem," but state it precisely and show that the hypothesis is clearly satisfied. Unless otherwise indicated, all references to measure and integration are in the sense of Lebesgue.

Part I. Choose 2 of the following 3 problems.

1. Let $\{A_n\}_{n\geq 1}$ be a sequence of Lebesgue measurable subsets of $[0,1]$. Assume that 1 is a limit point of the sequence $\{m(A_n)\}$, where m denotes the Lebesgue measure on $[0,1]$. Prove that there exists a subsequence $\{A_{n_k}\}_{k\geq 1}$ such that

$$m\left(\bigcap_{k=1}^{\infty} A_{n_k}\right) > 0.$$

2. Let $A \subset (0,1)$ be a measurable set and $m(A) = 0$. Show that

$$m(\{x^2 : x \in A\}) = 0 \quad \text{and} \quad m(\{\sqrt{x} : x \in A\}) = 0.$$

3. Suppose A and B are measurable subsets of $\mathbb{R}^n$, each one of strictly positive but finite measure. Prove that there exists a vector $\mathbf{c} \in \mathbb{R}^n$ such that

$$m\left((A + \mathbf{c}) \cap B\right) > 0.$$

[Hint: consider the outer measure of A and B. Or, alternatively, consider convolving characteristic functions related to A and B.]

Part II. Choose 2 of the following 3 problems.

4. Let m be the Lebesgue measure on $\mathbb{R}$. Let $\{f_n\}$, $\{g_n\}$, and $\{h_n\}$ be sequences of integrable functions on $\mathbb{R}$. Suppose that f, g, and h are such that

   (i) $f, h \in L^1(\mathbb{R})$,

   (ii) $\lim_n f_n(x) = f(x)$, $\lim_n g_n(x) = g(x)$, and $\lim_n h_n(x) = h(x)$, for a.e. x,

   (iii) $f_n(x) \leq g_n(x) \leq h_n(x)$ for a.e. x, and

   (iv) $\lim_n \int_{\mathbb{R}} f_n \, dm = \int_{\mathbb{R}} f \, dm$, and $\lim_n \int_{\mathbb{R}} h_n \, dm = \int_{\mathbb{R}} h \, dm$.

   Prove that
$$\lim_{n\to\infty} \int_{\mathbb{R}} g_n \, dm = \int_{\mathbb{R}} g \, dm$$

5. Prove the function
$$\phi(x) = \int_0^\infty \frac{t^{x-1}}{1+t} \, dt$$
   is continuous for $0 < x < 1$. Is $\phi(x)$ differentiable for $0 < x < 1$?

6. Let $p(\mathbf{x})$ be a polynomial in two variables (i.e. $\mathbf{x} = (x_1, x_2)$), but *not* the zero polynomial. Prove that the set of points $\mathbf{x} \in \mathbb{R}^2$ with $p(\mathbf{x}) = 0$ has 2-dimensional Lebesgue measure zero, i.e. $m(p^{-1}(0)) = 0$.

Part III. Choose 1 of the following 2 problems.

7. Let f be a measurable non-negative function on the measure space (X, Σ, μ), with $0 < \mu(X) < \infty$. Let
$$\|f\|_\infty = \sup\left\{M : \mu\left(f^{-1}(M-\delta, M]\right) > 0, \ \forall\, \delta > 0\right\}.$$
   Show that
$$\lim_{n\to\infty} \left(\int_X f(x)^n d\mu(x)\right)^{\frac{1}{n}} = \|f\|_\infty.$$

8. Suppose F is absolutely continuous on $[0,1]$ and that $g \in L^1([0,1])$, with $\int_0^1 g = 0$. Prove the "integration by parts" law:
$$\int_0^1 F(x)g(x)\,dx = -\int_0^1 \left[F'(x)\int_0^x g\right] dx.$$