Instructions: You must solve 2 problems from Part I, 2 problems from Part II, and 1 problem from Part III. All problems have equal weight. Each solution submitted must be written on a separate sheet of paper with your name and problem number at the top. Indicate on a separate sheet the problems you omit.

Carefully show all your steps. You may appeal to a well known theorem, but state it precisely and show that the hypothesis is clearly satisfied. Unless otherwise indicated, all references to measure and integration are in the sense of Lebesgue, and λ denotes Lebesgue measure on $\mathbb{R}$. Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

Part I. Choose 2 of the following 3 problems.

1. Let $\mathcal{P}(X)$ denote the class of all subsets of a non-empty set X. Suppose that μ is finitely additive measure on a field $\mathcal{U} \subseteq \mathcal{P}(X)$, and $\mu(X) < \infty$. Prove that μ is countably additive on $\mathcal{U}$ if and only if for each decreasing sequence

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$$

in $\mathcal{U}$, with $\cap_{n=1}^{\infty} A_n = \emptyset$, we have $\lim_{n\to\infty} \mu(A_n) = 0$.

2. Let $A \subset (0, 1)$ be a measurable set and $\lambda(A) = 0$. Show that

$$\lambda\{x^2 : x \in A\} = 0 \text{ and } \lambda\{\sqrt{x} : x \in A\} = 0.$$

3. Suppose that μ is a measure on $\mathcal{B}_1 := \mathcal{B}(\mathbb{R}^1)$ that is finite for bounded sets, and is translation-invariant, i.e., $\mu(B + x) = \mu(B)$ for all $B \in \mathcal{B}_1$ and $x \in \mathbb{R}^1$.

   (i) Show that $\mu(B) = \alpha\lambda(B)$ for some $\alpha \geq 0$.

   (ii) Does the above result extend to $\mathbb{R}^d$? Explain.

Part II. Choose 2 of the following 3 problems.

1. Let

$$f_n(x) = \frac{n}{x(\ln x)^n}$$

for $x \geq e$ and $n \in \mathbb{N}$.

   (i) For which $n \in \mathbb{N}$ does the Lebesgue integral $\int_e^\infty f_n(x)dx$ exist?

   (ii) Determine $\lim_{n\to\infty} f_n(x)$ for $x > e$.

   (iii) Does the sequence $\{f_n\}$ satisfy the assumptions of Lebesgue's dominated convergence theorem?

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2. For $n \geq 3$, let

$$f_n(x) = \begin{cases} n^2x & \text{if} \quad 0 \leq x < 1/n \\ \\ 2n - n^2x & \text{if} \ 1/n \leq x \leq 2/n \\ \\ 0 & \text{if} \ 2/n < x \leq 1. \end{cases}$$

   Sketch the graphs of f_3 and f_4. Prove that if g is a continuous real-valued function on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = g(0).$$

   (Hint: First show that $\int_0^1 f_n(x)dx = 1$.)

3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces given by $X = Y = [0,1]$;
   $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$; $\mu =$ Lebesgue measure on $[0,1]$ and $\nu =$ counting measure on $[0,1]$.
   If $D = \{(x,x) : x \in [0,1]\}$ is the diagonal in $X \times Y$, then find
   (i) $\int_Y \int_X 1_D d\mu\, d\nu$, $\int_X \int_Y 1_D d\nu\, d\mu$, and $\int_{X\times Y} 1_D d(\mu \times \nu)$.
   (Hint: To compute $\int_{X\times Y} 1_D d(\mu \times \nu) = (\mu \times \nu)(D)$, use the definition of $\mu \times \nu$.)
   (ii) Explain why Fubini's theorem fails in part (i).

Part III. Choose 1 of the following 2 problems.

1. (i) If F is absolutely continuous on $[0,1]$, then show that F is a function of bounded variation on $[0,1]$.

   (ii) Does the converse to (i) hold? If so, prove it. Otherwise, give a counterexample.

2. Suppose that μ and ν are σ-finite measures on a measurable space $(X, \mathcal{M})$ with $\nu \ll \mu$. Define $\rho = \mu + \nu$. If $\dfrac{d\nu}{d\rho} = f$, prove that $0 \leq f < 1$ μ a.e. and $\dfrac{d\nu}{d\mu} = f/(1-f)$.