Real Analysis Comprehensive/Qualifying Exam August 2021

The exam has three parts. You must turn in **two** problems from each of parts I and II and **one** problem from part III. All problems have the same weight. Only five problems will be graded. Make sure to have your name clearly written on the solution sheets. You can use "well known theorem" from the lecture notes or any standard book on measure and integration, **but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied**, otherwise you **will not get a full credit**.
If X is a set and $A \subseteq X$ then χ_A denotes the indicator functions of the set A. λ will denote the Lebesgue measure on $\mathbb{R}$ or a given subset of $\mathbb{R}$. λ^d will denote the Lebesgue measure on $\mathbb{R}^d$ or a subset thereof. $\overline{\mathbb{R}}$ denotes the set of extended real numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$. Let $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$ and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. If $\mathbf{X}$ is a Banach space then $\mathbf{X}^*$ denotes the dual space, ie., the space of all bounded linear maps $\varphi : \mathbf{X} \to \mathbb{C}$.

PART I

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1) Let $(X, \mathcal{A})$ be a measurable space and Y a set. For a function $f : Y \to X$ let

$$f^{-1}(\mathcal{A}) = \{f^{-1}(A) \mid A \in \mathcal{A}\}.$$

Show that $f^{-1}(\mathcal{A})$ is a σ-algebra on Y and that $f^{-1}(\mathcal{A})$ is the smallest σ-algebra such that $f : Y \to X$ is measurable.

2) Let $Q = [0,1]^d$ be the unit cube in $\mathbb{R}^d$ and assume that $f : Q \to \mathbb{R}$ is continuous. Show that the the graph of f

$$\Gamma(f) = \{(x, f(x)) \mid x \in Q\}$$

has measure zero.

3) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $f : X \to \overline{\mathbb{R}}_+$. Define $\mu_f : \mathcal{A} \to \overline{\mathbb{R}}_+$ by

$$\mu_f(A) = \int_A f d\mu = \int f \chi_A d\mu.$$

a) Show that μ_f is a measure.

b) Let $E = \{x \in X \mid f(x) < \infty\}$. Show that E is σ-finite with respect to μ_f.

4) Assume that X is a countable set and that μ is the counting measure. Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions and $f : X \to \mathbb{R}$. Show that $f_n \to f$ in measure if and only if $f_n \to f$ uniformly.

PART II

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5) Let $p(x, y)$ be a polynomial in two variables, but not the zero polynomial. Prove that the st

$$\{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0\} = p^{-1}(0) \subset \mathbb{R}^2$$

has a two-dimensional Lebesgue measure zero.

6) Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $f : X \to \mathbb{R}_+$ integrable. Suppose that for every $n \in \mathbb{N}$ we have

$$\int_X f(x)^n d\mu(x) = \int_X f(x) d\mu(x).$$

Show that $f(x) = \chi_A$ almost everywhere, where $A = \{x \in X \mid f(x) = 1\}$.

7) Let f be nonnegative Lebesgue measurable function on $(0, \infty)$ such that f^2 is integrable. Show the following

a) f is integrable on all finite intervals $(0, x]$, $x > 0$.

b) Define $F : (0, \infty) \to \mathbb{R}_+$ by

$$F(x) = \int_{(0,x]} f(x) d\lambda(x), \quad x > 0.$$

Show that $\lim_{x \searrow 0} \frac{F(x)}{\sqrt{x}}$ exists and equals to 0.

8) For $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = n \sin(x/n) \frac{1/x}{1 + x^2}.$$

a) Show that $f_n \in L^1(\mathbb{R}_+)$.

b) Determine the limit

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x) d\lambda(x)$$

if it exists.

PART III

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9) Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces. Assume that $T : \mathbf{X} \to \mathbf{Y}$ is a bounded linear map. For $\varphi \in Y^*$ define $T^\top(\varphi) : \mathbf{X} \to \mathbb{C}$ by

$$(T^\top(\varphi))(x) = \varphi(Tx).$$

Show that $T^\top(\varphi) \in \mathbf{X}^*$ and that $T^\top : \mathbf{Y}^* \to \mathbf{X}^*$ is linear and bounded with $\|T^\top\| \leq \|T\|$.

10) Let $\{f_n\}$ be a sequence in $L^1(X, \mu)$ and assume that there exists a measurable function f such that $f_n \to f$ uniformly.

a) If $\mu(X) < \infty$ then $f \in L^1(X, \mu)$ and $f_n \to f$ in L^1.

b) Give an example of a sequence $\{f_n\}$ in $L^1(\mathbb{R})$ such that $f_n \to f$ uniformly but the sequence $\{f_n\}$ does not converge to f in L^1.