PART I

1) Let $(X, A)$ be a measurable space and $Y$ a set. For a function $f : Y \to X$ let 

$$f^{-1}(A) = \{ f^{-1}(A) \mid A \in A \}.$$ 

Show that $f^{-1}(A)$ is a $\sigma$-algebra on $Y$ and that $f^{-1}(A)$ is the smallest $\sigma$-algebra such that $f : Y \to X$ is measurable.

2) Let $Q = [0, 1]^d$ be the unit cube in $\mathbb{R}^d$ and assume that $f : Q \to \mathbb{R}$ is continuous. Show that the graph of $f$ 

$$\Gamma(f) = \{(x, f(x)) \mid x \in Q \}$$ 

has measure zero.

3) Let $(X, A, \mu)$ be a $\sigma$-finite measure space and let $f : X \to \mathbb{R}_+$. Define $\mu_f : A \to \mathbb{R}_+$ by 

$$\mu_f(A) = \int_A f \, d\mu = \int f \chi_A d\mu.$$ 

a) Show that $\mu_f$ is a measure.

b) Let $E = \{ x \in X \mid f(x) < \infty \}$. Show that $E$ is $\sigma$-finite with respect to $\mu_f$.

4) Assume that $X$ is a countable set and that $\mu$ is the counting measure. Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions and $f : X \to \mathbb{R}$. Show that $f_n \to f$ in measure if and only if $f_n \to f$ uniformly.

PART II

5) Let $p(x, y)$ be a polynomial in two variables, but not the zero polynomial. Prove that the set 

$$\{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0\} = p^{-1}(0) \subset \mathbb{R}^2$$
has a two-dimensional Lebesgue measure zero.

6) Let \((X, \mathcal{A}, \mu)\) be a finite measure space and \(f : X \to \mathbb{R}_+\) integrable. Suppose that for every \(n \in \mathbb{N}\) we have

\[
\int_X f(x)^n d\mu(x) = \int_X f(x) d\mu(x).
\]

Show that \(f(x) = \chi_A\) almost everywhere, where \(A = \{x \in X \mid f(x) = 1\}\).

7) Let \(f\) be nonnegative Lebesgue measurable function on \((0, \infty)\) such that \(f^2\) is integrable. Show the following

a) \(f\) is integrable on all finite intervals \((0, x], x > 0\).

b) Define \(F : (0, \infty) \to \mathbb{R}_+\) by

\[
F(x) = \int_{(0,x]} f(x) d\lambda(x), \quad x > 0.
\]

Show that \(\lim_{x \searrow 0} \frac{F(x)}{\sqrt{x}}\) exists and equals to 0.

8) For \(n \in \mathbb{N} = \{1, 2, 3, \ldots\}\) define \(f_n : \mathbb{R} \to \mathbb{R}\) by

\[
f_n(x) = n \sin(x/n) \frac{1}{1 + x^2}.
\]

a) Show that \(f_n \in L^1(\mathbb{R}_+)\).

b) Determine the limit

\[
\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x) d\lambda(x)
\]

if it exists.

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**PART III**

9) Let \(X\) and \(Y\) be Banach spaces. Assume that \(T : X \to Y\) is a bounded linear map. For \(\varphi \in Y^*\) define \(T^\dagger(\varphi) : X \to \mathbb{C}\) by

\[
(T^\dagger(\varphi))(x) = \varphi(Tx).
\]

Show that \(T^\dagger(\varphi) \in X^*\) and that \(T^\dagger : Y^* \to X^*\) is linear and bounded with \(\|T^\dagger\| \leq \|T\|\).

10) Let \(\{f_n\}\) be a sequence in \(L^1(X, \mu)\) and assume that there exists a measurable function \(f\) such that \(f_n \to f\) uniformly.

a) If \(\mu(X) < \infty\) then \(f \in L^1(X, \mu)\) and \(f_n \to f\) in \(L^1\).

b) Give an example of a sequence \(\{f_n\}\) in \(L^1(\mathbb{R})\) such that \(f_n \to f\) uniformly but the sequence \(\{f_n\}\) does not converge to \(f\) in \(L^1\).