Qualifier Exam **Core Analysis** **August 2025**

There are twelve problems below in three parts.
Complete FIVE problems, doing at least ONE problem from each part.
All problems have the same weight of twenty points. Please mark the problems you want to be graded, and make sure to have your name clearly written on the solution sheets. You can use "well known theorems" from lectures or any standard book on measure and integration, but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied, otherwise you might not get full credit.

Part A:

(1) Suppose f is nonnegative and integrable on $\mathbb{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x : f(x) > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} f(x)dx = \int_0^\infty m(E_\alpha)d\alpha.$$

(2) Compute (and justify any convergence theorems used) the limit

$$\lim_{n\to\infty} \int_1^\infty \sin\left(\frac{x}{n}\right) \frac{n^3}{1+n^2x^2}dx.$$

(3) Recall the definition of the maximal function: for $f : \mathbb{R}^d \to \mathbb{R}$ integrable,

$$f^*(x) := \sup_{x\in B} \frac{1}{m(B)} \int_B |f(y)|dy,$$

where the supremum is taken over all balls B containing the point x. Prove that, if $f \in L^1$ and f is not identically zero, there exists $C > 0$ so that for all $|x| > 1$ we have

$$f^*(x) \geq \frac{C}{|x|^d}.$$

(4) Prove that $L^\infty(\mathbb{R})$ is not separable. That is, any dense subset of $L^\infty(\mathbb{R})$ must be uncountable.

Part B:

(1) Show that every open subset U of $\mathbb{R}$ can be written as a countable disjoint union $\cup_{n\in\mathbb{N}}(a_n, b_n)$ of open intervals.

(2) Let E be a measurable set in $[0,1]$. Assume that, for every open interval $I \subset [0,1]$ we have

$$m(E \cap I) \geq \frac{1}{2} m(I).$$

Prove that $m(E) = 1$.

(3) Prove that, for every Lebesgue integrable function f on $\mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(t)\sin(nt)dt = 0.$$

HINT: Do it for step functions.

(4) Let E have finite measure. Let $\{f_k\}$ be a sequence of measurable functions on E converging pointwise to a measurable function f.

a) State Egorov's Theorem for f_k and f.

b) Let $\{g_k\}$ be a sequence of measurable functions defined on a set E with $m(E) < \infty$. Suppose that, for every $x \in E$, there is $M_x \in [0, \infty)$ such that $|g_k(x)| \leq M_x$ for all k. For any $\epsilon > 0$, show that there exists a closed $F \subseteq E$ and a finite M such that $m(E \setminus F) < \epsilon$ and $|g_k(x)| \leq M$ for all k and all $x \in F$. We do not assume that g_k converges in any way.

Part C:

(1) Let $f(x) = x^2 \sin(1/x)$ and $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$, with $f(0) = g(0) = 0$.

a) Show that f and g are everywhere differentiable.

b) Show that f is of bounded variation on the interval $[-1, 1]$, but g is not.

(2) Suppose F is absolutely continuous on $[0,1]$ and $g \in L^1([0,1])$ with $\int_0^1 g(y)dy = 0$. Prove that

$$\int_0^1 F(x)g(x)dx = -\int_0^1 \left(F'(x) \int_0^x g(y)dy \right) dx.$$

(3) Let μ and ν be two nonnegative finite measures on $(X, \mathcal{A})$.

a) Give the definitions for "ν absolutely continuous w.r.t. μ" and "ν and μ are mutually singular".

b) If ν is both absolutely continuous and singular to μ, prove that $\nu(E) = 0$ for all $E \in \mathcal{A}$.

(4) Let Ω be an uncountable set and let $\mathcal{A}$ be the collection of all subsets $A \subset \Omega$ such that either A or A^C is at most countable. Define

$$\mu(A) := \begin{cases} 0, & \text{if } A \text{ is at most countable,} \\ 1, & \text{if } A^C \text{ is at most countable.} \end{cases}$$

Show that $\mathcal{A}$ is a σ-algebra and that μ is a measure on $\mathcal{A}$.