

Analysis Qualifier Examination for August 2026

Twelve problems are listed below in three parts. Complete five problems, doing at least one from each part.

All problems are weighted twenty points each. Please indicate the problems to be graded and write your name clearly on the solution sheets. You can apply theorems from the lecture or standard text on measure and integral, but make sure to state the hypothesis and conclusions of the theorem you are applying and clearly argue that the hypothesis of the theorem is satisfied to get full marks.

Part A

1. Given an interval $[a, b] \in \mathbb{R}$ show that a monotone decreasing function defined on the interval can have at most a countable number of jumps.
2. If f is continuous on $[a, b]$ and has a bounded derivative on (a, b) show that f is of bounded variation on $[a, b]$. Is boundedness of f' necessary for f to be of bounded variation? Justify your answer.
3. Let f be a nonnegative Lebesgue integrable function on $\mathbb{R}$. Show that the function

$$F(x) = \int_{-\infty}^x f(t) dt,$$

is continuous.

4. Let $f_n : [0, \infty) \rightarrow \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \leq n, \\ 0 & \text{if } x > n. \end{cases}$$

- (a) Show f_n converges uniformly on $\mathbb{R}$, but

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx \neq \int_0^\infty \left(\lim_{n \rightarrow \infty} f_n(x) \right) dx.$$

- (b) State the Dominated Convergence Theorem and explain why the sequence of functions does not contradict it.

Part B

1. Let f be an integrable function on $[0, 1]$ and suppose that for every bounded continuous function $g(x)$ on $[0, 1]$ that

$$\int_0^1 f(x)g(x) dx = 0.$$

Show $f = 0$ almost everywhere.

2. If $f(x)$ is integrable on $\mathbb{R}$ is it true that $\lim_{n \rightarrow \infty} \int_{-\infty}^\infty f(x) \sin(nx) dx = 0$? Prove or give counter example.
3. If $f \in L^1[a, b]$ show

$$\frac{1}{2r} \int_{x-r}^{x+r} f(t) dt = f(x), \text{ a.e. in } (a, b).$$

Why is this called the Lebesgue differentiation theorem?

4. Let f be defined by $f(0) = 0$ and $f(x) = x \sin(1/x)$, for $x \neq 0$. Is f of bounded variation on $[-1, 1]$?

Part C

1. Let f be defined and bounded for $Q = \{(x, t); 0 \leq x \leq 1, 0 \leq t \leq 1\}$, suppose f is Lebesgue measurable in x for each t and for each x , $\frac{\partial f((x, t))}{\partial t}$ exists and is bounded on Q . Show

$$\frac{\partial}{\partial t} \int_0^1 f(x, t) dx = \int_0^1 \frac{\partial}{\partial t} f(x, t) dx.$$

2. Show if f is integrable on a Lebesgue measurable set E and

$$\left(\int_E |f(x)|^p dx \right)^{1/p} = \|f\|_p, \quad 1 \leq p < \infty.$$

Then $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$, where $\|f\|_\infty = \text{ess sup}\{|f(x)| \mid x \in E\}$.

3. Given the finite measure space $([0, 1], \mathcal{C}, \mathcal{L}^1)$, where $\mathcal{C}$ is the σ -algebra of Borel subsets of $[0, 1]$ and $\mathcal{L}^1$ denotes the Lebesgue measure on $\mathbb{R}$. Give an example of a measure λ defined on $\mathcal{C}$ and its Lebesgue decomposition with respect to $\mathcal{L}^1$.
4. Give an example of a function defined on $[0, 1]$ that is of bounded variation and continuous but not absolutely continuous and explain why.