Analysis Qualifier Examination for Spring 2026

Twelve problems are listed below in three parts. Complete five problems, doing at least one from each part.

All problems are weighted twenty points each. Please indicate the problems to be graded and write your name clearly on the solution sheets. You can apply theorems from the lecture or standard text on measure and integral, but make sure to state the hypothesis and conclusions of the theorem you are applying and clearly argue that the hypothesis of the theorem is satisfied to get full marks.

Part A

1. Let f be defined and bounded for $Q = \{(x,t);\, 0 \le x \le 1,\, 0 \le t \le 1\}$, suppose f is Lebesgue measurable in x for each t and for each x, $\frac{\partial f((x,t)}{\partial t}$ exists and is bounded on Q. Show
$$\frac{\partial}{\partial t}\int_0^1 f(x,t)\,dx = \int_0^1 \frac{\partial}{\partial t} f(x,t)\,dx.$$

2. Let $E \subset \mathbb{R}$ be a measurable set with the property that for every open interval I one has $m(E \cap I) \le \frac{1}{2} m(I)$. Show $m(E) = 0$.

3. If f is continuous on $[a,b]$ and has a bounded derivative on (a,b) show that f is of bounded variation on $[a,b]$. Is boundedness of f' necessary for f to be of bounded variation? Justify your answer.

4. Let f be a nonnegative Lesbegue integrable function on $\mathbb{R}$. Show that the function
$$F(x) = \int_{-\infty}^{x} f(t)\,dt,$$
is continuous.

Part B

1. Given an interval $[a,b] \in \mathbb{R}$ show that a monotone increasing function defined on the interval can have at most a countable number of jumps.

2. If $f(x)$ is integrable on $\mathbb{R}$ is it true that $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\cos(nx)\,dx = 0$? Prove or give counter example.

3. Given the collection $\{f_n(x)\}_{n=1}^{\infty}$ with $f_n(x) : \mathbb{R} \to \mathbb{R}$ Lebesgue measurable. Show the set
$$S = \{x;\, \lim_{n\to\infty} f_n(x)\,,\ \text{exists}\},$$
is Lebesgue measurable.

4. Let f be defined by $f(0) = 0$ and $f(x) = x^2 \sin(1/x^2)$, for $x \neq 0$. Is f of bounded variation on $[-1,1]$?

Part C

1. Show if f is integrable on a Lebesgue measurable set E and

$$(\int_E |f(x)|^p \, dx)^{1/p} = ||f||_p, \ 1 \leq p < \infty.$$

   Then $\lim_{p\to\infty} ||f||_p = ||f||_\infty$, where $||f||_\infty = ess\ sup\{|f(x)|\, x \in E\}$.

2. Given the finite measure space $([0,1], \mathcal{C}, \mathcal{L}^1)$, where $\mathcal{C}$ is the σ-algebra of Borel subsets of $[0,1]$ and $\mathcal{L}^1$ denotes the Lebesgue measure on $\mathbb{R}$. Give an example of a measure λ defined on $\mathcal{C}$ and its Lebesgue decomposition with respect to $\mathcal{L}^1$.

3. Give an example of a function defined on $[0,1]$ that is of bounded variation and continuous but not absolutely continuous and explain why.

4. If $f \in L^1[a,b]$ show

$$\frac{1}{2r} \int_{x-r}^{x+r} f(t)\, dt = f(x), \text{ a.e. in } (a,b).$$

   Why is this called the Lebesgue differentiation theorem?