Real Analysis Comprehensive/Qualifying Exam - January 2023

The exam has three parts. You must turn in two problems from each of **Part I** and **Part II** and one problem from **Part III**. All problems have the same weight of twenty points. Only five problems will be graded and counted towards the final grade. Please mark the problems you want to be graded, and make sure to have your name clearly written on the solution sheets.

Notation: $(X, \mathcal{A}, \mu)$ will always stand for a measure space. If nothing else is said, then the σ-algebra on $\mathbb{R}$, $\mathbb{R}^d$, $d \geq 2$, or subsets of these sets will be the Borel σ-algebra, and the measure is the Lebesgue measure λ (in case $d = 1$) (and, respectively, λ^d, $d > 1$).

You can use "well known theorems" from the lecture notes or any standard book on measure and integration, but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied, otherwise you will not get full credit.

PART I

(1) Let E be a measurable set in $[0,1]$ and let $c > 0$. If $\lambda(E \cap I) \geq c\lambda(I)$ for all open intervals $I \subset [0,1]$, show that $\lambda(E) = 1$.

(2) Suppose $f_n : X \to \mathbb{R}$ is a measurable function for each $n \in \mathbb{N}$, where $(X, \mathcal{A}, \mu)$ is a measure space. Prove that the set

$$S = \{x : \lim_{n\to\infty} f_n(x) \,\text{exists}\}$$

is a measurable set.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function that is not almost everywhere infinite. Prove that there exists a subset $S \subset \mathbb{R}$ of positive measure such that f is bounded on S.

(4) Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 1/n, & \text{if } |x| \leq n, \\ 0, & \text{if } |x| > n. \end{cases}$$

(a) Show that f_n converges to 0 uniformly on $\mathbb{R}$, and that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = 2,$$

while

$$\int_{\mathbb{R}} \left(\lim_{n\to\infty} f_n\right)(x)dx \neq 2.$$

(b) Explain why this example does not contradict the Lebesgue Dominated Convergence Theorem.

(5) Let $(X, \mathcal{A}, \mu)$ be a measure space for which $\mu(X) < \infty$. Show

$$L^q(X, d\mu) \subseteq L^p(X, d\mu)$$

whenever $1 \leq p \leq q \leq \infty$..

PART II

(6) Provide an example of a sequence $\{f_n\}$ of measurable functions on $[0, 1]$ such that $f_n \to f$ almost everywhere, and $f_n \geq 0$, yet

$$\liminf_{n\to\infty} \int_0^1 f_n d\lambda \neq \int_0^1 f d\lambda.$$

(7) Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} (\sin(nt))\, f(t)\, dt = 0$$

for every Lebesgue integrable function f on $\mathbb{R}$. (Hint: Do the problem first for step functions.)

(8) Suppose that we have two decompositions of a measure space for a *signed measure* μ, as in the Hahn decomposition theorem. That is,

$$X = P \cup N = P' \cup N'$$

where each measurable subset of P or P' has non-negative measure, and each measurable subset of N or N' has non-positive measure. Prove that every measurable subset of

$$(P \Delta P') \cup (N \Delta N')$$

is a μ-null set (i.e. has μ measure 0), where $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

(9) Let μ and ν be non-negative finite measures on $(X, \mathcal{A})$. If $\nu \perp \mu$ and $\nu \ll \mu$, prove that $\nu = 0$, the identically zero measure on $\mathcal{A}$.

PART III

(10) Prove that if f is a real-valued Lebesgue integrable function on $\mathbb{R}$, then

$$\lim_{h\to 0} \int_{\mathbb{R}} |f(x + h) - f(x)| dx = 0.$$

(11) Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose that $(f_n)_{n\geq 1}$ is a sequence of non-negative integrable functions on X that converges almost everywhere to an integrable function f, and

$$\lim_{n\to\infty} \int_X f_n dx = \int_X f dx.$$

Show that for all $B \in \mathcal{A}$, $\lim_{n\to\infty} \int_B f_n d\mu = \int_B f d\mu$. (Hint: Use Fatou's lemma.)