Qualifier Exam **Core Analysis** **January 2024**

There are twelve problems below in three parts.
Complete FIVE problems, doing at least ONE problem from each part.
All problems have the same weight of twenty points. Please mark the problems you want to be graded, and make sure to have your name clearly written on the solution sheets. You can use "well known theorems" from lectures or any standard book on measure and integration, but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied, otherwise you might not get full credit.

Part A:

(1) Let $f(x)$ be a real-valued measurable function on a finite measure space $(X, \mathcal{M}, \mu)$. Show that

$$\lim_{n\to\infty} \int_X \cos^{2n}(\pi f(x))d\mu(x) = \mu\left(\{x : f(x) \in \mathbb{Z}\}\right).$$

(2) Let $f \in L^1(\mathbb{R})$ and let E be a bounded measurable set. Let x_n be a sequence in $\mathbb{R}$ with $|x_n| \to \infty$ as $n \to \infty$. Show that

$$\lim_{n\to\infty} \int_{x_n+E} f(y)dy = 0.$$

(3) Prove that, for every Lebesgue integrable function f on $\mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} (\sin(nt))f(t)dt = 0.$$

HINT: Do it first for step functions.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be nonnegative and integrable. Prove that, for almost every $a \in \mathbb{R}$,

$$F(a) := \int_{\mathbb{R}} \frac{f(x)}{|x-a|^{1/2}}dx < \infty.$$

HINT: Since $F \geq 0$, it suffices to show that $\int_{[-R,R]} F(a)da < \infty$ for any R.

Part B:

(1) Let E be a measurable set in $[0,1]$ and let $c > 0$. If $m(E \cap I) \geq cm(I)$ for all open intervals $I \subset [0,1]$, show that $m(E) = 1$.

(2) Suppose A and B are Lebesgue-measurable subsets of $\mathbb{R}^d$, each one of strictly positive but finite measure. Prove that there exists a vector $c \in \mathbb{R}^d$ such that $m((A+c) \cap B) > 0$.
HINT: Consider the outer measure of A and B. Or, for a much quicker proof, consider the convolution of the indicator functions $\chi_{-A} * \chi_B$.

(3) Let X be a space and $\mathcal{M}$ a σ-algebra. Let μ and ν be two measures defined on $(X, \mathcal{M})$. Suppose there is a sequence $A_n \in \mathcal{M}$ such that

$$\lim_{n\to\infty} \mu(A_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \nu(X \setminus A_n) = 0.$$

Prove that μ and ν are mutually singular.

(4) Let F be a measurable subset of $\mathbb{R}$. Let $E \subset F$ consist of those points $x \in F$ for which there exists a $\delta > 0$ with $m((x-\delta, x+\delta) \cap F) = 0$. Show that $m(E) = 0$.

Part C:

(1) For $f : [0,\infty) \to \mathbb{R}$, the Laplace transform $\mathcal{L}[f] : [0,\infty) \to \mathbb{R}$ is given by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)dt.$$

a) Prove that $\mathcal{L}[f * g] = \mathcal{L}[f]\mathcal{L}[g]$. Here $(f * g)(x) = \int_0^x f(x-y)g(y)dy$.
b) Prove that, if $f \in L^p([0,\infty))$, then $|\mathcal{L}[f](s)| \leq Cs^{(1-p)/p}$. Here $1 \leq p \leq \infty$.

(2) Recall the definition of the maximal function: for $f : \mathbb{R}^d \to \mathbb{R}$ integrable,

$$f^*(x) := \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)|dy,$$

where the supremum is taken over all balls B containing the point x. Prove that, if $f \in L^1$, and f is not identically zero, then for some $C > 0$ and all $|x| > 1$,

$$f^*(x) \geq \frac{C}{|x|^d}.$$

(3) Let $1 < p < q < r < \infty$. If $f \in L^p(\mathbb{R})$ and $f \in L^r(\mathbb{R})$, then $f \in L^q(\mathbb{R})$.

(4) Show that $L^p[0,1]$ is separable for $1 \leq p < \infty$ (i.e., there exists a countable dense subset), and that $L^\infty[0,1]$ is not separable.