MATHEMATICS COMPREHENSIVE EXAMINATION

CORE I – Analysis

August 2002

Directions: This test consists of four parts (A), (B), (C), and (D). You must do all problems in Parts (A) and (B), choose two problems from Part (C), and choose two problems from Part (D). Please answer the problems in the order they appear and turn in only those problems you wish to have graded. You have two and a half hours for this test.

Part (A). Answer "true" or "false". There is no need to justify the answer.

1. Let (X, d) be a complete metric space. Suppose ρ is another metric on X and generates the same topology as d. Then (X, ρ) is also a complete metric space.
2. The series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly to a continuous function on $\mathbb{R}$.
3. If f is a Lipschitz function on a finite closed interval $[a, b]$, then f is differentiable almost everywhere with respect to the ordinary Lebesgue measure.
4. The orthogonal complement of any subspace of a Hilbert space is always a closed subspace.
5. For any measure space (X, μ), convergence in $L^1(\mu)$ always implies convergence almost everywhere with respect to μ.
6. Let f be a Lebesgue integrable function on $[a, b]$. Then the function $g(x)$ defined by $g(x) = \int_a^x f(t)\, dt$ is differentiable at every point $x \in (a, b)$.
7. If a Banach space B is separable, then its dual space B^* is also separable.

Part (B). Give a counterexample to each problem to show that the statement is false. There is no need to justify the answer.

8. If f is a continuous function of bounded variation on $[0, 1]$, then f is absolutely continuous on $[0, 1]$.
9. If f is Lebesgue integrable on $[0, 1]$, then f^2 is also Lebesgue integrable on $[0, 1]$.
10. If a metric space (X, d) is separable, then it is a complete metric space.

Part (C). Choose two of the four problems.

C1. Define $T_n(x) = \cos(n \arccos x)$, $n = 1, 2, \ldots$. Show that the functions $T_n(x)$ satisfy the recursion formula $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0$, $n \geq 1$, and are polynomials.

C2. Let f be a uniformly continuous function on the open interval $(0, 1)$. Prove that the right-hand limit $\lim_{x \to 0^+} f(x)$ exists.

C3. Let f and g be continuous functions on $\mathbb{R}$. Suppose $f = g$ almost everywhere with respect to the ordinary Lebesgue measure. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

C4. Show that the function $f(x) = \dfrac{\sin x}{x}$ is not Lebesgue integrable on $(0, \infty)$.

Part (D). Choose two of the four problems.

D1. Find the limit $\displaystyle\lim_{n \to \infty} \int_0^1 \frac{n}{1 + n^2 x^2}\, dx$.

D2. Derive the Fourier series expansion of the function

$$f(x) = \begin{cases} -1, & \text{if } -\pi < x \leq 0; \\ 1, & \text{if } 0 < x \leq \pi. \end{cases}$$

with respect to the orthonormal basis $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \ldots, \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}, \ldots \right\}$ for the Hilbert space $L^2[-\pi, \pi]$.

D3. Check whether the set $\{\sin(nx);\ n = 1, 2, 3, \ldots\}$ of functions is equi-continuous on the interval $[0, 1]$.

D4. Check whether the function defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } 0 < x \leq 1; \\ 0, & \text{if } x = 0. \end{cases}$$

is a function of bounded variation.