# Mathematics Comprehensive Examination

## Core 1 - Analysis

August 2003

**Directions:** This test consists of three parts (A), (B), and (C). You must do *all eight problems in part (A).* Choose *two problems from Part (B)*, and also *two from part (C).* Please answer the problems in the order in which they appear and turn in only those that you wish to have graded. You have two and a half hours for this test. Good Luck!

**Terminology:** The integrals that appear in this examination are to be understood as Lebesgue integrals. Lebesgue measure on the real line is denoted by $\lambda$. The symbol $I_A$ denotes the indicator (or characteristic) function of the set $A$. The space $C[0,1]$ is equipped with the supremum norm.

**Part A:** Answer *True or False. If false, give a simple reason or a counterexample.*

1. If $\{f_n\}$ is a sequence of Lebesgue measurable functions on $(0,1)$ such that
   (i) $|f_n| \leq g \in L^1(0,1)$ for all $n$, and
   (ii) $f_n \to f$ in *measure* as $n \to \infty$,
   *then* $\displaystyle\lim_{n\to\infty} \int_{(0,1)} |f_n - f| d\lambda = 0$
2. Let $A \subset C[0,1]$ be the subset consisting of all differentiable functions in the interval $(0,1)$. Then, $A$ is complete.
3. Let $\{f_\alpha : \alpha \in A\}$ be a family of measurable functions indexed by an arbitrary index set $A$. Define $g = \sup_{\alpha\in A} f_\alpha$. Then $g$ is a measurable function.
4. The closed algebra generated by the two functions $f_1(x) = 1$, and $f_2(x) = x^2$, is $C[0,1]$.
5. The unit sphere $\{f : \max\limits_{t\in[0,1]} |f(t)| \leq 1\}$ in the space $C[0,1]$ is compact.
6. If the $L^2[0,1]$-norm of a function $f$ is equal to a large number $N$, then the $L^1[0,1]$-norm of $f$ is at most $\sqrt{N}$.
7. The function $f(x) = x\sin(1/x)$ for $0 < x < 1$ is a function of bounded variation.
8. If the sequence $\{f_n\}$ converges to $f$ in $L^p(\mathbf{R})$, and the sequence $\{g_n\}$ converges to $g$ in $L^q(\mathbf{R})$ where $1/p + 1/q = 1$, then the sequence $\{f_n g_n\}$ converges to $fg$ in $L^1(\mathbf{R})$.

**Part B:**

1. Consider the function $f_n(x) = \dfrac{n}{e^x + n^2 x}$ for $x \in [0,1]$ and $n \geq 1$. Evaluate $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$. Justify your answer.

2. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $[0,1]$ such that

$$\sum_{n=1}^{\infty} \lambda(\{x \in [0,1] : |f_n(x)| > 1\}) < \infty.$$

   Prove that $-1 \leq \lim_{n\to\infty} \inf f_n(x) \leq \lim_{n\to\infty} \sup f_n(x) \leq 1$ for almost every $x \in [0,1]$.

3. Let $E$ be a subset of a separable metric space $(X,d)$. Prove that $(E,d)$ is a separable metric space.

4. Let $f$ be an integrable function over $\mathbf{R}$. Show that given $\epsilon > 0$, there exists a $\delta > 0$ such that for every $A \subset \mathbf{R}$ with $\lambda(A) < \delta$, the following holds:

$$\int_A f\,d\lambda < \epsilon$$

**Part C:**

1. Let $\{f_n\}$ be a sequence of real-valued functions defined on a compact metric space $(K,d)$ such that $\{f_n\}$ is pointwise bounded and equicontinuous.
   (a) Show that $\{f_n\}$ is uniformly bounded.
   (b) Prove that $g(x) = \inf\{f_n(x) : n \geq 1\}$ is a uniformly continuous function.

2. Let $a$ denote the sequence $\{a_n\}$ of real numbers. Define $c_0 = \{a : \lim_{n\to\infty} a_n = 0\}$. Show that $c_0$ is a Banach space when equipped with the $||.||_\infty$ norm. (Note: $||a||_\infty = \sup_n |a_n|$).

3. Let $X$ and $Y$ be Banach spaces. Let $\{T_n\}$ be a sequence of bounded linear operators from $X$ to $Y$ such that $\{T_n x\}$ is Cauchy for each $x \in X$.
   (a) Prove that there exists a bounded linear operator $T$ such that $\lim_{n\to\infty} ||T_n x - Tx|| = 0$ for each $x$.
   (b) Prove or disprove (by giving a counter-example): $\lim_{n\to\infty} ||T_n - T|| = 0$.

4. Let $f \in L^p((0,1])$ for some $p \geq 1$. Let $J_i = (\frac{i-1}{n}, \frac{i}{n}]$ for $i = 1, 2, \cdots, n$. Define
   $f_n(t) = n \int_{J_i} f(x)\,dx \quad \text{if} \quad t \in J_i$
   where $i = 1, 2, \cdots, n$. Show that $||f_n||_{L^p} \leq ||f||_{L^p}$ for all $n$.