Directions: *Write your solutions on the blank paper provided, in the order in which they appear on the exam. (Leave a blank page so you can come back to insert a problem in order if you temporarily skip one.) Both problems in **Part A** are required. Also pick any three of the five in **Part B**.* Thus exactly five problems are to be handed in. All integrals are understood in the sense of Lebesgue. You have $2\frac{1}{2}$ hours. Good Luck!

Part A. *Do each of the following two problems.*

1. Let $A \subset (0,1)$ be a measurable set and $m(A) = 0$. Prove that $m\{\sqrt{x} : x \in A\} = 0$.
2. Let $1 < p < q < r < \infty$. If $f \in L^p(\mathbb{R}) \cap L^r(\mathbb{R})$, prove that $f \in L^q(\mathbb{R})$.

Part B. *Do any three of the following five problems.*

3. Prove that the set of real-valued functions that are of bounded variation on the closed interval $[a, b]$ forms a real vector space. That is, show it is closed under addition and scalar multiplication of functions.
4. Prove that, if the *outer* measure $m^*(E) < \infty$ and if there exist intervals $I_1, \dots, I_n$ such that $m^*\left(E \triangle \bigcup_{i=1}^{n} I_i\right) < \infty$, then each of the intervals I_i is finite.
5. Let $1 \le p < \infty$. Suppose $f_n \in L^p[0,1]$, $\|f_n\|_p \le 1$, and $f_n \to f$ a.e..
   (a) Prove that $f \in L^p[0,1]$ and $\|f\|_p \le 1$.
   (b) If $1 < p < \infty$ and $g \in L^q[0,1]$ with $\frac{1}{p} + \frac{1}{q} = 1$ then $\int_0^1 f_n g \to \int_0^1 fg$ as $n \to \infty$. (Hint: Use Egoroff's theorem to identify a set E on which f_n converges uniformly and $m(\tilde{E})$ small. Apply Hölder's Inequality.)
6. Let $f \in L^1(0, \infty)$, and suppose that $\int_0^\infty x|f(x)|\,dx < \infty$. Prove that the function
   $$g(y) = \int_0^\infty e^{-xy} f(x)dx$$
   is differentiable at every $y \in (0, \infty)$.
7. A real-valued function f on an interval I for which there exists a constant C such that
   $$|f(x) - f(y)| \le C|x - y|$$
   for all x and y in I is called a *Lipschitz function.*
   (a) Prove that every Lipschitz function is absolutely continuous.
   (b) Let f be absolutely continuous on an interval. Prove that f is Lipschitz if and only if f' is essentially bounded.
