Mathematics Comprehensive Examination
Core 1: Analysis
August 2004

Instructions: Do the *starred problem (#4)* and three of the remaining four. *Hand in solutions to a **total of four** problems.* Each problem submitted for evaluation must be written on a separate sheet of paper with your name at the top. Each problem will be scored out of 25 points. Partial credit will be given, but completeness, conciseness, and clarity are important. You should quote theorems and other well known results that you use. Make sure they are properly used. All references to measure and integration refer to Lebesgue measure and integration on the real line. You have $2\frac{1}{2}$ hours to complete this exam. We wish you well.

1. The following lemma finds uses in various approximation results. Prove it.

   Lemma Suppose f and g are continuous nonnegative functions on $[a,b]$, $-\infty < a, b < \infty$. Let $Z(f) = \{x \in [a,b] : f(x) = 0\}$ be the set of zeroes of f. Suppose $Z(g) \subset Z(f)$. Then, for each $\epsilon > 0$, there exists a constant $M > 0$ such that

   $$f(x) \leq \epsilon + Mg(x),$$

   for all $x \in [a,b]$.

   Hint: Use the fact that $[a,b]$ is compact

2. Let $C[0,1]$ be the space of continuous functions on the interval $[0,1]$. For $f \in C[0,1]$ and each $n = 1, 2, \ldots$ let $T_n f$ be defined by

   $$T_n f(x) = \int_0^1 n s^n f(sx)\, ds, \qquad x \in [0,1].$$

   Prove that $T_n f \to f$ uniformly.

   Hint: First verify for $f(t) = t^m$, then f a polynomial, and finally the general case.

3.
   - Evaluate: $k = \lim_{n\to\infty} \int_0^\pi |\sin(nx)|\, dx$.
   - Prove: $\lim_{n\to\infty} \int_a^b |\sin(nx)|\, dx = \frac{k}{\pi}(b-a)$.
   - Let f be an integrable function on $\mathbb{R}$. Prove:

   $$\lim_{n\to\infty} \int_{-\infty}^{\infty} |\sin(nx)|\, f(x)\, dx = \frac{k}{\pi} \int_{-\infty}^{\infty} f(x)\, dx.$$

   Hint: For the third part begin with f a step function.

4. * Suppose $f \in L^1[0,1]$. Define $F : (0,\infty) \to \mathbb{R}$ by

$$F(x) = \int_0^1 e^{-xt} f(t)\, dt.$$

Show for $0 < x < \infty$ that

$$F'(x) = \int_0^1 -te^{-xt} f(t)\, dt.$$

Hint: $F'(x) = \lim_{n\to\infty} \frac{F(x+h_n)-F(x)}{h_n}$ where h_n is a sequence that converges to 0 and $0 < x + h_n$.

5. Let $E \subset \mathbb{R}$ be a measurable set with the property that

$$m(E \cap I) \leq \frac{m(I)}{2},$$

for every open interval I. Prove that $m(E) = 0$.

Hint: Extend the inequality to an open set and then to an arbitrary measurable set.