MATHEMATICS COMPREHENSIVE EXAMINATION

CORE 1 - Analysis

January 2003

Directions: This test consists of three parts (A), (B), and (C). You must do all 10 problems in part (A), choose two problems from Part (B), and also two from part (C). Please answer the problems in the order they appear and turn in only those that you wish to have graded. You have two and a half hours for this test.

Terminology: The integrals that appear in this examination are to be understood as Lebesgue integrals. The Lebesgue measure on the real line is denoted by λ. The symbol I_A denotes the indicator (ie. characteristic) function of the set A. The space $C[0,1]$ is equipped with the supremum norm.

Part A: Answer True or False. If false, give a simple reason, or a counterexample.

1. The Lebesgue monotone convergence theorem holds for any decreasing sequence $\{f_n\}$ of non-negative Lebesgue measurable functions.
2. For almost every (with respect to the Lebesgue measure) real number in $[0,1]$, the number 8 appears in its decimal expansion.
3. Closed subset of a complete metric space is complete.
4. The sequence of functions $\{f_n\}$ defined by $f_n(x) = x\, I_{\{n\,\leq x\,\leq n+1/n\}}$ converges to 0 almost uniformly as $n \to \infty$.
5. Any continuous function defined on $(0,1]$ can be approximated pointwise by a sequence of polynomials.
6. Let c denote the subspace of l^∞ of all sequences $(x_1, x_2, \cdots)$ for which $\lim_{n\to\infty} x_n$ exists. Then c is a closed subset of l^∞.
7. For any decreasing sequence $\{B_n\}$ of non-empty, closed, bounded sets in a Banach space X, $\cap_{n\geq 1} B_n$ is non-empty.
8. If $f \in L^p \cap L^r$ where $1 \leq p < r \leq \infty$, then $f \in L^s$ for any $s \in [p, r]$.
9. The space l^∞ is a separable metric space.
10. Let $D \subset C[0,1]$ be the set of continuously differentiable functions equipped with the supremum norm. For $f \in D$, let $Tf = f'$. Then T is a bounded linear operator.

Part B:

1. Let $f_n(x) = \dfrac{e^{-x/n}}{n}$ for $x \geq 0$ and $n = 1, \cdots$

(a) Show that $f_n \in L^1[0, \infty)$ for all n.

(b) Are the hypotheses of the Lebesgue dominated convergence theorem (LDCT) satisfied for $\{f_n\}$? Explain.

(c) Let $h_n = n\sin(x/n)$ for $x \in [0,1]$, and $n = 1, \cdots$ Are the hypotheses of LDCT satisfied for $\{h_n\}$? Explain.

2. Let E be a Lebesgue measurable set in $\mathbf{R}^1$. Show that given $\epsilon > 0$, there exists an open set G such that $E \subset G$ and $\lambda(G - E) < \epsilon$.

3. Let $f(x) = \int_{[0,\infty)} e^{-(x-y)^2} (1 + y^2)^{-1} \, dy$. for all $x > 0$. Find the derivative $f'(x)$. Justify the steps involved.

4. Prove or disprove: The function $f(x) = \sin x + \cos x$ is Lebesgue integrable on $\mathbf{R}^1$.

Part C:

1. Let $X = C[0,1]$. Let $Y = \{g : g(x) = \int_{[0,1]} \cos(x + f^2(t))dt, \;\; x \in [0,1], \quad \text{and} \quad f \in X\}$. Show that closure of Y is compact in X.

2. Show that any continuous function $f \in C([0,1] \times [0,1])$ can be approximated uniformly by functions of the form $\phi(x,y) = \sum_{j=1}^{k} g_j(x)h_j(y)$ for some k with $g_j, \; h_j \in C[0,1]$ for all j.

3. Let $f(x) = x\sin(x^{-1})\cos(x^{-1})$ if $x \in (0,1]$, and $f(0) = 0$. Determine whether f is a function of bounded variation on $[0,1]$.

4. (a) State the open-mapping theorem.

(b) Let X be a closed subspace of both $C[0,1]$ and $L^2[0,1]$. Show that there is a constant C such that $||f||_2 \leq ||f||_\infty$ and $||f||_\infty \leq C||f||_2$ for all $f \in X$.