MATHEMATICS COMPREHENSIVE EXAMINATION

CORE I — Analysis

January 2005

Directions: This exam has three parts. **Part A** consists of short answers, and you should do all of them. In **Part B** you should do three of the six proofs, and from **Part C**, choose one of the two. Write your solutions on the blank paper provided, in the order in which they appear in the exam. All integrals and measures of sets are understood in the sense of Lebesgue in the context of the real number system. You have $\mathbf{2\frac{1}{2}}$ **hours** to complete this exam. Good luck!

Part A. Do all of the problems in this part. All answers are to be short and concise.

1. State the definitions of the following concepts.

i. The Borel sets of real numbers.

ii. Convergence in measure of a sequence of real-valued functions.

2. Give complete and precise statements of the following results.

i. The Riesz Representation Theorem for the spaces L^p, $1 \leq p < \infty$.

ii. The Hahn-Banach Theorem on extensions of linear functionals in normed linear spaces.

iii. The Open-Mapping Theorem, concerning operators between Banach spaces.

3. Suppose $f_n \to f$ a.e. on a measurable set E. In precisely what sense is the convergence "almost uniform"?

4. Provide short concise justifications of the following statements.

i. In a Hilbert space, $x_n \to x$ and $y_n \to y$ implies $(x_n, y_n) \to (x, y)$.

ii. Let $p + q = pq$. For $g \in L^q(E)$, define $\hat{g} \in (L^p(E))^*$ as $\hat{g}(f) = \int_E gf$. Prove that $\|\hat{g}\| = \|g\|_{L^q(E)}$.

iii. Any set from $\mathbb{R}$ of outer measure zero is measurable.

Part B. Prove three of the following six statements.

1. The vector space of finite sequences is dense in ℓ^p for $1 \le p < \infty$, but it is not dense in ℓ^∞.

2. $L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

3. Let H be a Hilbert space with inner product $(\cdot, \cdot)$.

i. $2(x, y) \le \|x\|^2 + \|y\|^2$ for all $x, y \in H$ with equality if and only if $x = y$.

ii. Let $\{z_1, \ldots, z_n\}$ be an orthonormal set in H. Then $\| \sum_{i=1}^n \alpha_i z_i \| = (\sum_{i=1}^n |\alpha_i|^2)^{1/2}$.

4. If $f_n \to f$ pointwise and $\|f_n\|_p \le 1$ $(1 \le p \le \infty)$, then $\|f\|_p \le 1$.

5. If f is absolutely continuous on $[0, 1]$, then the total variation of f on $[0, 1]$ is equal to $\int_0^1 |f'|$.

6. **i.** Let A be an arbitrary subset of $\mathbb{R}$. There exists a measurable set E such that $A \subset E$ and $\mathrm{m}(E) = \mathrm{m}^*(A)$.

ii. There exist disjoint subsets A_1 and A_2 of $\mathbb{R}$ such that $\mathrm{m}^*(A_1) + \mathrm{m}^*(A_2) \ne \mathrm{m}^*(A_1 \cup A_2)$.

Part C. Prove one of the following two.

1. Suppose that f is a real-valued function of bounded variation on $[0, 1]$. Then

i. f has a right-hand limit at each point of $[0, 1)$ and left-hand limit at each point of $(0, 1]$.

ii. The set of points of discontinuity of f is countable.

iii. If, in addition to being of bounded variation on $[0, 1]$, f is also absolutely continuous on $[0, T]$ for each $T \in (0, 1)$, then there exists an absolutely continuous function g on $[0, 1]$ that coincides with f on $[0, 1)$.

2. Let X and Y be Banach spaces, and let $\mathcal{L}(X, Y)$ be the Banach space of bounded linear operators from X to Y endowed with the operator norm. Let x be a fixed element of X and y^* a fixed element of the dual Y^* of Y. Consider the bilinear mapping $\phi : X \times Y^* \to [\mathcal{L}(X, Y)]^* :: (x, y^*) \mapsto \phi_{x,y^*}$, where ϕ_{x,y^*} is defined by $\phi_{x,y^*}(A) = y^*(Ax)$ for all $A \in \mathcal{L}(X, Y)$. Then

i. $\phi_{x,y^*} = 0$ if and only if $x = 0$ or $y^* = 0$;

ii. $\|\phi_{x,y^*}\| = \|x\| \|y^*\|$.