Instructions: *Do all five problems. Each problem submitted for evaluation must be written on a separate sheet of paper with your name at the top. Each problem will be scored out of* 20 *points. Your answers must be complete, concise, and clear. You must show that the hypotheses of well known results that you use are satisfied. Make sure they are properly used. All references to measure and integration are in the sense of Lebesgue. Your have* $2\frac{1}{2}$ *hours to complete this exam. We wish you well.*

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1. Let $C_{0,0}[0,1]$ be the space of all continuous real functions f on the interval $[0,1]$ satisfying $f(0) = f(1) = 0$. Let $\mathcal{P}_{0,0}$ be the subspace of polynomials in $C_{0,0}[0,1]$. Show that $\mathcal{P}_{0,0}$ is dense in $C_{0,0}[0,1]$ in the sup norm.

2. If E is a measurable subset of $[0,1]$ then there is a measurable subset $A \subset E$ such that $m(A) = \frac{1}{2}m(E)$.

3. Find and justify the limits:

   (a) $\displaystyle \lim_{n\to\infty} \int_0^n \frac{\sin x}{1+nx^2}\,dx$

   (b) $\displaystyle \lim_{n\to\infty} \int_0^{e^n} \frac{x}{1+nx^2}\,dx.$

4. Let $f_n \to f$ in L^p, that is $\lim_{n\to\infty} ||f_n - f||_p = 0$ and $g_n \to g$ in L^q where $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that

$$f_n g_n \to fg \quad \text{in } L^1.$$

5. Suppose $f, g \in L^1[0,1]$, $f > 0$, $g > 0$, and $fg \ge 1$. Prove

$$\int_0^1 f \int_0^1 g \ge 1$$