Instructions: Do ***two (2)*** *problems from* ***Part A*** and ***three (3)*** *from* ***Part B***, making **a total of five (5)** problems. *Start each chosen problem on a fresh sheet of paper.* **Write your name on each sheet at the top**. Be sure to cite all theorems that you apply, checking that all hypotheses are satisfied. You have 3 hours for this test. If you have time, **please check your work *carefully***. Clarity is important. If you find an error but are unsure how to fix it in the time allotted, say so, because it is better if you recognize it! When finished, *clip your papers together in numerical order of the problems chosen.*

Notation: The symbol l denotes Lebesgue measure on either $\mathbb{R}$ or $\mathbb{R}^n$ unless otherwise indicated, and $(X, \mathfrak{A}, \mu)$ is an abstract measure space. A *field* or *σ-field* is also called an *algebra* or *σ-algebra* of sets, respectively. Integration is in the sense of Lebesgue, not that of Riemann, unless stated to the contrary.

Part A: Measures. Do Two (2) problems.

1. For each example, if true *explain briefly* why, or if false, *describe briefly* a counterexample.
   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.
      i. The pre-image of a Lebesgue measurable set under f must be measurable.
      ii. The pre-image of a Borel set under f must be measurable.
   (b) In $(X, \mathfrak{A}, \mu)$, suppose $\mu(X) < \infty$. Let f_n be a sequence of measurable functions.
      i. If f_n converges in measure then it converges almost everywhere.
      ii. If f_n converges almost everywhere then it converges in measure.

2. Suppose that the measures λ, μ, ν on a σ-field $\mathfrak{A} \subset \mathfrak{P}(X)$ have the relationship $\lambda \prec \mu \prec \nu$ where λ and μ are finite and ν is σ-finite. Prove that $\lambda \prec \nu$ and that the equation

$$\frac{d\lambda}{d\nu} = \frac{d\lambda}{d\mu}\frac{d\mu}{d\nu}$$

   is satisfied by the three Radon-Nikodym derivatives. (Hint: Use an increasing sequence of special simple functions to approximate $\frac{d\lambda}{d\mu}$.)

3. Suppose $f : X \to \mathbb{R}^*$, the extended real number system, is a measurable function that has *finite values almost everywhere* on the measure space $(X, \mathfrak{A}, \mu)$ where $\mu(X) > 0$. Prove that there is a measurable set of *strictly positive* measure on which f is bounded.

4. Let $A \subset (0, 1)$ be a measurable set and $m(A) = 0$. Prove that
   (a) $l\left\{x^2 \mid x \in A\right\} = 0$
   (b) $l\left\{\sqrt{x} \mid x \in A\right\} = 0$.

Part B: Integrals. Do three (3) problems.

5. Let f be an integrable real-valued function on a measure space $(X, \mathfrak{A}, \mu)$.

   (a) If $\epsilon > 0$, prove that there exists $\delta > 0$ such that $A \in \mathfrak{A}$ and $\mu(A) < \delta$ implies
$$\left| \int_A f \, d\mu \right| < \epsilon.$$

   (b) Now suppose $f \in L^1(\mathbb{R}, \mathfrak{L}, l)$ where l is Lebesgue measure, and prove that
$$\int_b^\infty f \, dl \to 0$$
   as $b \to \infty$.

6. Let $f \in L^1(0, \infty)$, and suppose that $\int_0^\infty x|f(x)| \, dx < \infty$. Prove that the function
$$g(y) = \int_0^\infty e^{-xy} f(x) dx$$
   is differentiable at every $y \in (0, \infty)$ and find the value of $g'(y)$.

7. Prove that, if f is a real-valued Lebesgue-integrable function on $\mathbb{R}$, then
$$\lim_{x \to 0} \int_{\mathbb{R}} |f(x+t) - f(t)| \, dl(t) = 0.$$

8. Let $\nu(E)$ to be the number of elements in E for each $E \subseteq \mathbb{N}$, the set of all natural numbers, so that that ν is σ-finite, as is $\nu \times \nu$. (Integrations with respect to these measures are summations.) Let
$$f(m,n) = \begin{cases} 2 - 2^{-m} & \text{if } m = n \\ -2 + 2^{-m} & \text{if } m = n+1 \\ 0 & \text{if } m \notin \{n, n+1\}. \end{cases}$$
   Show that
$$\int_{\mathbb{N}} \int_{\mathbb{N}} f(m,n) \, d\nu(m) \, d\nu(n) \neq \int_{\mathbb{N}} \int_{\mathbb{N}} f(m,n) \, d\mu(n) \, d\nu(m)$$
   and explain why this does not violate Fubini's theorem.

9. Prove that, if f is absolutely continuous on $[0,1]$, then the total variation of f on $[0,1]$ is equal to $\int_0^1 |f'| \, dl$.