Instructions: Solve ***five (5)*** problems, including ***at least two (2) from Part A*** and ***at least two (2)*** *from* ***Part B***. *Start each chosen problem on a fresh sheet of paper.* **Write your name and the problem number on each sheet at the top. Turn in exactly 5 problems—no more and no fewer—even if the solutions are imperfect.** *Clip your papers together in numerical order of the problems chosen when finished.* Please check your work carefully: Logical exposition matters. You have 3 hours. **Good luck!**
Symbols: Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^n$ is l or l^n respectively, $(X, \mathfrak{A}, \mu)$ is an abstract measure space, and 1_S is the indicator (or characteristic) function of the set S.

Part A: Measures & Measurable Functions

1. Let $f : X \to \mathbb{R}^*$ be an extended real valued measurable function on a finite measure space $(X, \mathfrak{A}, \mu)$. Suppose that $f(x)$ is finite for almost all x.
   (a) Prove that for each $\epsilon > 0$ there exists a set $A \in \mathfrak{A}$, with $\mu(X \setminus A) < \epsilon$, such that f is bounded on A.
   (b) Give an example of a measurable function $f : [0, 1] \to \mathbb{R}$ for which it is impossible that f be bounded on A, if $X \setminus A$ is a μ-null set. *Justify* your choice.

2. Consider the set $I = \mathbb{R} \setminus \mathbb{Q}$ of all irrational numbers.
   (a) Prove that I is not an F_σ-set. (You may assume the Baire Category theorem, but explain how its hypotheses are satisfied.)
   (b) Prove that I is a G_δ-set.
   (c) True or False: There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous at x if and only if $x \in \mathbb{Q}$. Explain briefly—full proof not required.

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable. Prove that f is equal almost everywhere to a *Borel* measurable function h. (Hint: The function f is the pointwise limit of a sequence $\phi_n \in \mathfrak{S}$, the set of simple functions. Modify the functions ϕ_n suitably.)

4. Suppose A and B are measurable subsets of $\mathbb{R}$, each one of strictly positive but finite measure. Prove that there exists a number $c \in \mathbb{R}$ such that

   $$l((A + c) \cap B) > 0.$$

   Hints: Consider the outer measure of A and B. Or, alternatively, consider the convolution $1_{-A} * 1_B(x) = \int_{\mathbb{R}} 1_{-A}(x - t) 1_B(t)\, dl(t)$.

Part B: Integrals

5. Consider the sequence of functions $f_n(x) = 1_{[-n,n]}(x) \sin\left(\frac{\pi x}{n}\right)$, for all $x \in \mathbb{R}$.

   (a) Determine $f(x) = \lim_{n\to\infty} f_n(x)$, the pointwise limit, and show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{R}$. Does the sequence converge uniformly on $\mathbb{R}$?

   (b) Show that $\int_{\mathbb{R}} f(x)\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx$.

   (c) Are the assumptions of Lebesgue's dominated convergence theorem satisfied? Prove your answer.

6. Prove that $\lim_{\alpha\to\infty} \int_{\mathbb{R}} f(t) \sin \alpha t \, dl(t) = 0$, for every Lebesgue integrable function f on $\mathbb{R}$. (Hint: Give a proof first for $f(x) = 1_{[a,b]}(x)$, the indicator function of an interval.)

7. Suppose $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by $h(x, y) = x - y$. Let $E \subset \mathbb{R}$ be a *Borel* set such that $l(E) = 0$. Prove that the plane measure $l^2\left(h^{-1}(E)\right) = 0$. (Hint: Use Fubini's Theorem. Justify that the hypotheses of Fubini's Theorem are satisfied.)

8. (a) Let f be an integrable real-valued function on a measure space $(X, \mathfrak{A}, \mu)$. If $\epsilon > 0$, prove that there exists $\delta > 0$ such that $A \in \mathfrak{A}$ and $\mu(A) < \delta$ implies

   $$\left| \int_A f \, d\mu \right| < \epsilon.$$

   (b) Suppose that a finite nonnegative measure λ is absolutely continuous with respect to Lebesgue measure l on $\mathbb{R}$: That is, $\lambda \prec l$. *Define*

   $$F(x) = \lambda\big((-\infty, x)\big),$$

   for all $x \in \mathbb{R}$. Prove that F is an absolutely continuous *function* on $\mathbb{R}$. (Hint: You may use the Radon-Nikodym theorem, and also the result of part (a).)