Instructions: All problems have equal weight. You are asked to submit 6 problems in total and each problem submitted must be written on a separate sheet of paper with your name and problem number at the top. Follow the instructions for each part carefully. Unless otherwise indicated all references to measure and integration are in the sense of Lebesgue on the real line. You have three hours. We wish you well!

Part I: Determine the validity of each of the following statements; simply write True or False. You do not have to justify your answers. Assume measure and integration on the real line in the sense of Lebesgue.

1. (a) Suppose $f$ is a measurable function and $E$ is a measurable set. Then $f^{-1}(E)$ is a measurable set.
   (b) Suppose $f_n$ converges to $f$ a.e. on a set of finite measure. Then $f_n$ converges to $f$ in measure.
   (c) A bounded measurable function defined on a set of finite measure is integrable.
   (d) If $f \in L^\infty[0,1]$ and $\epsilon > 0$ then there is a continuous function $g$ such that
       \[ \|f - g\|_\infty < \epsilon. \]
   (e) If $f$ is of bounded variation and $f' = 0$ a.e. then $f$ is a constant a.e.

Part II: Answer 2 of the following 3 problems. For part (a) carefully state the requested result or definition. For part (b) give a short explanation why your example works.

2. (a) State Fatou’s lemma for a complete measure space $(X, \mathcal{A}, \mu)$.
   (b) Give an example of a sequence $f_n$ of nonnegative functions on $\mathbb{R}$ for which
       \[ \int_{\mathbb{R}} f < \liminf \int_{\mathbb{R}} f_n, \]
       where $f$ is the pointwise limit of $f_n$. (note the strict inequality)

3. (a) State the Lebesgue dominated convergence theorem for a complete measure space $(X, \mathcal{A}, \mu)$.
   (b) Give an example of a sequence $f_n$ of nonnegative integrable functions on $[0,1]$ that converges pointwise to an integrable function $f$ and for which
       \[ \lim_{n \to \infty} \int_0^1 f_n \neq \int_0^1 f. \]

4. (a) State the definition of convergence in measure on a measure space $(X, \mathcal{A}, \mu)$.
   (b) Give an example of a sequence $(f_n)$ of bounded measurable functions on $[0,1]$ which converge to a function $f$ in measure but does not converge pointwise at any $t \in [0,1]$. 
Part III: Answer 3 of the following 5 problems. Your rigorous proofs must be complete and clear. Make sure that well known results are properly used and quoted. This includes showing that the hypotheses are satisfied.

5. If $f$ is absolutely continuous on an interval $(a, b)$, $E \subset (a, b)$, and $m(E) = 0$ then

$$m(f(E)) = 0.$$ 

Here $m$ denotes Lebesgue measure and $f(E) = \{ f(x) : x \in E \}$.

6. Suppose $f$ is in $L^4[0,1]$. Show that

$$\int_0^1 \frac{f(x)}{x^4} \, dx$$

is finite.

7. Determine

$$\lim_{n \to \infty} \int_1^\infty \sin \left( \frac{x}{n} \right) \frac{n^3}{1 + n^2 x^3} \, dx.$$ 

Justify your answer.

8. If $E$ is a measurable subset of $[0,1]$ prove that there is a measurable subset $A \subset E$ such that $m(A) = \frac{1}{2} m(E)$. Here $m$ denotes Lebesgue measure.

9. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and $m(X) = 1$. Suppose $f$ and $g$ are positive measurable functions on $X$ such that $fg \geq 1$. Show that

$$\int_X f \, d\mu(x) \int_X g \, d\mu(x) \geq 1.$$