Instructions: All problems have equal weight. You are asked to submit 6 problems in total and each problem submitted must be written on a separate sheet of paper with your name and problem number at the top. **Follow the instructions for each part carefully.** *Unless otherwise indicated* all references to measure and integration are in the sense of Lebesgue on the real line. You have three hours. We wish you well!

Part I: Determine the validity of each of the following statements; simply write True or False. You do not have to justify your answers. Assume measure and integration on the real line in the sense of Lebesgue.

1. (a) Suppose f is a measurable function and E is a measurable set. Then $f^{-1}(E)$ is a measurable set.

   (b) Suppose f_n converges to f a.e. on a set of finite measure. Then f_n converges to f in measure.

   (c) A bounded measurable function defined on a set of finite measure is integrable.

   (d) If $f \in L^\infty[0,1]$ and $\epsilon > 0$ then there is a continuous function g such that

   $$\|f - g\|_\infty < \epsilon.$$

   (e) If f is of bounded variation and $f' = 0$ a.e. then f is a constant a.e.

Part II: Answer 2 of the following 3 problems. For part (a) carefully state the requested result or definition. For part (b) give a short explanation why your example works.

2. (a) State Fatou's lemma for a complete measure space $(X, \mathfrak{A}, \mu)$.

   (b) Give an example of a sequence f_n of nonnegative functions on $\mathbb{R}$ for which

   $$\int_{\mathbb{R}} f < \liminf \int_{\mathbb{R}} f_n ,$$

   where f is the pointwise limit of f_n. (note the strict inequality)

3. (a) State the Lebesgue dominated convergence theorem for a complete measure space $(X, \mathfrak{A}, \mu)$.

   (b) Give an example of a sequence f_n of nonnegative integrable functions on $[0,1]$ that converges pointwise to an integrable function f and for which

   $$\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f.$$

4. (a) State the definition of convergence in measure on a measure space $(X, \mathfrak{A}, \mu)$.

   (b) Give an example of a sequence (f_n) of bounded measurable functions on $[0,1]$ which converge to a function f in measure but does not converge pointwise at any $t \in [0,1]$.

Part III: Answer 3 of the following 5 problems. Your rigorous proofs must be complete and clear. Make sure that well known results are properly used and quoted. This includes showing that the hypotheses are satisfied.

5. If f is absolutely continuous on an interval (a, b), $E \subset (a, b)$, and $m(E) = 0$ then
$$m(f(E)) = 0.$$
Here m denotes Lebesgue measure and $f(E) = \{f(x) : x \in E\}$.

6. Suppose f is in $L^4[0, 1]$. Show that
$$\int_0^1 \frac{f(x)}{x^{\frac{1}{4}}}\, dx$$
is finite.

7. Determine
$$\lim_{n\to\infty} \int_1^\infty \sin\left(\frac{x}{n}\right) \frac{n^3}{1 + n^2x^3}\, dx.$$
Justify your answer.

8. If E is a measurable subset of $[0, 1]$ prove that there is a measurable subset $A \subset E$ such that $m(A) = \frac{1}{2}m(E)$. Here m denotes Lebesgue measure.

9. Suppose $(X, \mathfrak{A}, \mu)$ is a measure space and $m(X) = 1$. Suppose f and g are positive measurable functions on X such that $fg \geq 1$. Show that
$$\int_X f\, d\mu(x) \int_X g\, d\mu(x) \geq 1.$$