Real Analysis **Comprehensive/Qualifying Exam** **January 2011**

Instructions: All problems have equal weight. Each problem submitted must be written on a separate sheet of paper with your name and problem number at the top. Your solutions must be complete and clear. You must show that the hypotheses of well known results that you use are satisfied. Make sure they are properly used and quoted. *Unless otherwise indicated* all references to measure and integration are in the sense of Lebesgue on the real line. You have three hours. We wish you well!

Part I: Choose 2 of the following 3 problems

1. Let $(X, \mathfrak{A}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ a measurable function (in the sense that $f^{-1}(-\infty, a) \in \mathfrak{A}$ for all $a \in \mathbb{R}$). Prove $f^{-1}(B)$ is measurable for every Borel set B in $\mathbb{R}$.

2. Let $(X, \mathfrak{A}, \mu)$ be a measure space and suppose $f_n : X \to \mathbb{R}$ is a measurable function for each $n \in \mathbb{N}$. Prove that the set
$$\left\{x \,\middle|\, \lim_{n\to\infty} f_n(x) \text{ exists}\right\}$$
is a measurable set.

3. Suppose A and B are Lebesgue measurable subsets of $\mathbb{R}$, each one of strictly positive but finite measure. Prove that there exists a number $c \in \mathbb{R}$ such that
$$\ell((A + c) \cap B) > 0,$$
where ℓ denotes Lebesgue measure. (Hints: Consider the outer measure of A and B. Or, alternatively, consider the convolution $1_{-A} * 1_B$, and use Fubini's theorem.)

Part II: Choose 2 of the following 3 problems

1. Prove that
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} (\sin nt) f(t)\, dt = 0$$
for all Lebesgue integrable functions f on $\mathbb{R}$. (Hint: The compactly supported step functions are dense in $L^1(\mathbb{R})$.)

2. Let $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x > 0 \\ 1 & \text{if } x = 0. \end{cases}$ Prove that $\lim_{b\to\infty} \int_0^b f(x)\, dx$ exists but that $f \notin L^1[0, \infty)$.

3. Define $f(x) = \int_{\mathbb{R}} \cos(xy) g(y)\, dy$ for $x \in \mathbb{R}$ and g an integrable function on $\mathbb{R}$. Show that f is continuous.

Part III: Choose 1 of the following 2 problems

4. Show that if f is absolutely continuous on $[a,b]$, then the total variation $V_a^x f$, where $x \in [a,b]$ is given by

$$V_a^x f = \int_a^x \left|f'(t)\right| \, dt.$$

(Hint: Let $F(x) = V_a^x f$. First show $F'(x) \geq |f'(x)|$ a.e. by using $V_x^{x+h} \geq |f(x+h) - f(x)|$ and then conclude $\int_a^x |f'(t)| \, dt \leq \int_a^x F'(t) \, dt \leq V_a^x f$. Then show the reverse inequality.)

5. Let X be the set of Lebesgue measurable functions on $[0,1]$. For $f, g \in X$ define

$$\rho(f,g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, dx.$$

Let $f_n \in X$, $n \in \mathbb{N}$. Show that $\lim\limits_{n\to\infty} \rho(f_n, f) = 0$ if and only if $f_n \to f$ in measure.