Instructions: Do five (5) of the 8 problems, including at least two (2) from Part A and at least two (2) from Part B. Start each chosen problem on a fresh sheet of paper. Write your name at the top of each sheet. Please turn in 5 problems, even if the solutions are imperfect, for partial credit. Logical rigor is important. Describe which theorems you apply, checking that all hypotheses are satisfied. If you see a gap in your proof but not how to fix it, state this because it is better that you recognized it! Clip your papers together in numerical order of the problems chosen when finished. You have 3 hours. Good luck!

Symbols: Integration is in the sense of Measure and Integration theory. Lebesgue measure on \( \mathbb{R} \) or \( \mathbb{R}^n \) is \( l \), and \( (X, \mathcal{A}, \mu) \) is an abstract measure space. Also, \( 1_S \) is the characteristic, or indicator, function of a set \( S \).

Part A: Measures

1. If \( E \) is a Lebesgue measurable subset of \([0,1]\) prove carefully that there is a measurable subset \( A \subset E \) such that \( l(A) = \frac{1}{2}l(E) \).

2. Let \( f : X \to \mathbb{R}^* \) be an extended real-valued measurable function on a finite measure space \( (X, \mathcal{A}, \mu) \). Suppose that \( f(x) \) is finite for almost all \( x \).
   
   (a) Prove that for each \( \epsilon > 0 \) there exists a set \( A \in \mathcal{A} \), with \( \mu(X \setminus A) < \epsilon \), such that \( f \) is bounded on \( A \).
   
   (b) Give an example for which it is impossible to require that \( X \setminus A \) be a \( \mu \)-null set.

3. Suppose \( f_n : X \to \mathbb{R} \) is a measurable function for each \( n \in \mathbb{N} \), where \( (X, \mathcal{A}, \mu) \) is a measure space. Prove that the set \( S = \left\{ x \mid \lim_{n \to \infty} f_n(x) \text{ exists} \right\} \) is a measurable set.

4. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be Lebesgue measurable. Prove that \( f \) is equal almost everywhere to a Borel measurable function \( h \). Hint: The function \( f \) is the pointwise limit of a sequence \( \phi_n \in \mathcal{S} \), the set of simple functions. Modify the functions \( \phi_n \) suitably.
Part B: Integrals

5. Consider the sequence of functions \( f_n(x) := 1_{[-n,n]}(x) \sin \left( \frac{\pi x}{n} \right) \) for all \( x \in \mathbb{R} \).

(a) Determine \( f(x) = \lim_{n \to \infty} f_n(x) \) and show that the sequence \((f_n)_{n \in \mathbb{N}}\) converges uniformly on compact subsets of \( \mathbb{R} \). Does the sequence converge uniformly on \( \mathbb{R} \)?

(b) Show that \( \int_{-\infty}^{\infty} f(x) \, dx = \lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x) \, dx \). Are the assumptions of Lebesgue’s dominated convergence theorem satisfied?

6. Let \( f(x) \) be a real-valued measurable function on a finite measure space \((X, \mathcal{A}, \mu)\). Show that \( \lim_{n \to \infty} \int_X \cos^{2n}(\pi f(x)) \, d\mu = \mu\{x \mid f(x) \in \mathbb{Z}\} \) where \( \mathbb{Z} \) is the set of integers.

7. Suppose \( f \) and \( h \) are in \( L^1(\mathbb{R}) \), and let \( H(x, y) = f(x - y)h(y) \). Show that \( H \in L^1(\mathbb{R}^2) \).

Use this to show the function \( f * h = \int_{\mathbb{R}} f(x - y)h(y) \, dy \) is defined almost everywhere and is an integrable function on \( \mathbb{R} \). Then show that \( \|f * h\|_1 \leq \|f\|_1 \|h\|_1 \). (Hint: Use Fubini’s Theorem. You may assume the measurability of \( H \) over the plane.)

8. Do either part (a) or part (b) but not both: Credit given for one part only!

(a) Suppose that both \( f \) and \( \frac{\partial f}{\partial y} \) lie in \( L^1([a, b] \times [c, d]) \) and that \( f(x, y) \) is absolutely continuous as a function of \( y \) for almost all fixed values of \( x \). Prove \( \frac{d}{dy} \int_a^b f(x, y) \, dl(x) \)

exists and equals \( \int_a^b \frac{\partial f}{\partial y} (x, y) \, dl(x) \) for almost all \( y \). Hint: Prove that \( g \) is constant where \( g(y) = \int_a^b f(x, y) \, dl(x) - \int_c^y \left( \int_a^b \frac{\partial f}{\partial t}(x, t) \, dl(x) \right) \, dt \).

(b) Let \( \phi : [0, 1] \to [0, 1] \) be the Cantor function. Let \( f(x, y) = \phi(x + y)1_D(x, y) \) for all \((x, y) \in D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\} \). Show that \( \frac{d}{dy} \int_{\mathbb{R}} f(x, y) \, dl(x) \neq \int_{\mathbb{R}} \frac{\partial}{\partial y} f(x, y) \, dl(x) \). Reconcile this with the claim in part (a).