Instructions: Solve one of the problems (1)–(2), two of the problems (3)–(6) and two of the problems (7)–(10). Only turn in the solution to at most **five** problems. Turn in all your work even if you do not finish a problem. You might get partial credit. Make sure that you have written your name on all pages that you turn in.

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1. Let $\mathcal{A}$ be a σ-algebra on a set X and let $\mu : \mathcal{A} \to [0, \infty]$ be a finitely additive measure. Assume that $\mu(X) < \infty$. Show that the following are equivalent.

   (a) μ is countably additive.

   (b) For all decreasing sequences $\{A_j\}$ in $\mathcal{A}$ we have
   $$\mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} \mu(A_j)\,.$$

   (c) For all increasing sequences $\{B_j\}$ in $\mathcal{A}$ we have
   $$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} \mu(A_j)\,.$$

2. Let $(X, \mathcal{A}, \mu)$ be finite measure space and let $f : X \to \overline{\mathbb{R}}$ be a non-negative measureable function. For $n \in \mathbb{N}$ let $E_n = \{x \in X \mid n - 1 \leq f(x) < n\}$. Let $1 \leq p < \infty$. Show that $f \in L^p(X)$ if and only if $\displaystyle\sum_{n=1}^{\infty} n^p \mu(E_n) < \infty$.

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3. Let $f_n(x) = \dfrac{x}{n}\mathbf{1}_{[0,n]}(x)$.

   (a) Determine the limit $\displaystyle\lim_{n\to\infty} f_n$.

   (b) Show that $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\lambda \neq \int_{\mathbb{R}} \lim_{n\to\infty} f_n d\lambda$.

4. Find and justify the limits

   (a) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}} \mathbf{1}_{[0,n]}(x) \frac{\sin x}{1 + nx^2}\, d\lambda(x)$.

   (b) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}} \mathbf{1}_{[0,e^n]} \frac{x}{1 + nx^2}\, d\lambda(x)$.

5. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence of bounded measurable functions $f_n : X \to \mathbb{R}$. Assume that f_n converges uniformly to a function f. Show that $f_n \to f$ in L^1 and that $\displaystyle\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu$.

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6. Suppose that $f \in L^1(\mathbb{R})$ is such that $\int_{\mathbb{R}} |xf(x)|\, d\lambda(x) < \infty$. Show that

$$F(t) = \int_{\mathbb{R}} f(x)\cos(tx)\, d\lambda(x)$$

is differentiable and find $F'(t)$.

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7. Let $f \in L^1[0,1]$, $f \geq 0$. Show that $\sqrt{\int_{[0,1]} f\, d\lambda(x)} \geq \int_{[0,1]} \sqrt{f(x)}\, d\lambda(x)$.

8. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Let σ be the counting measure on $\mathbb{N}$. Finally let $Z = \mathbb{N} \times X$, $\mathcal{B} = \mathcal{P}(\mathbb{N}) \otimes \mathcal{A}$ and $\eta = \sigma \otimes \mu$.

   (a) Show that $E \subset Z$ is measurable if and only if $E_n = \{x \in X \mid (n,x) \in E\}$ is measurable for each $n \in \mathbb{N}$ and in that case $\eta(E) = \sum_{n=1}^{\infty} \mu(E_n)$.

   (b) For $f : Z \to \overline{R}$ and $n \in \mathbb{N}$ define $f_n : X \to \overline{R}$ by $f_n(x) = f(n,x)$. Then f is measurable if and only if f_n is measurable for all n.

   (c) Let the notation be as in (2) and assume that f is measurable. Then $f \in L^1(Z, \eta)$ if and only if $f_n \in L^1(X, \mu)$ for all $n \in \mathbb{N}$ and $\displaystyle\sum_{n=1}^{\infty} \int_X |f_n(x)|\, d\mu(x) < \infty$.

9. For $-\infty < a < b < \infty$ let $I = [a,b]$ and let μ be a signed measure on I (with respect to the Borel σ-algebra). Let $f(x) = \mu([a,x])$. Then f is of bounded variation.

10. For $1 \leq p < \infty$ let ℓ^p be the space of sequences $\{a_n\}_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} |a_n|^p < \infty$. Thus $\ell^p = L^p(\mathbb{N})$ with respect to the counting measure. For $1 \leq p \leq q < \infty$ show that $\ell^p \subset \ell^q$ and that $\|\{a_n\}\|_q \leq \|\{a_n\}\|_p$.