Instructions: Solve one of the problems (1)–(2), two of the problems (3)–(6) and two of the problems (7)–(10). Only turn in the solution to at most five problems. Turn in all your work even if you do not finish a problem. You might get partial credit. Make sure that you have written your name on all pages that you turn in.

1. Let $\mathcal{A}$ be a $\sigma$-algebra on a set $X$ and let $\mu : \mathcal{A} \to [0, \infty]$ be a finitely additive measure. Assume that $\mu(X) < \infty$. Show that the following are equivalent.

(a) $\mu$ is countably additive.

(b) For all decreasing sequences $\{A_j\}$ in $\mathcal{A}$ we have

$$\mu \left( \bigcap_{j=1}^\infty A_j \right) = \lim_{j \to \infty} \mu(A_j).$$

(c) For all increasing sequences $\{B_j\}$ in $\mathcal{A}$ we have

$$\mu \left( \bigcup_{j=1}^\infty B_j \right) = \lim_{j \to \infty} \mu(B_j).$$

2. Let $(X, \mathcal{A}, \mu)$ be finite measure space and let $f : X \to \mathbb{R}$ be a non-negative measurable function. For $n \in \mathbb{N}$ let $E_n = \{ x \in X \mid n - 1 \leq f(x) < n \}$. Let $1 \leq p < \infty$. Show that $f \in L^p(X)$ if and only if $\sum_{n=1}^\infty n^p \mu(E_n) < \infty$.

3. Let $f_n(x) = \frac{x}{n} 1_{[0,n]}(x)$.

(a) Determine the limit $\lim_{n \to \infty} f_n$.

(b) Show that $\lim_{n \to \infty} \int_\mathbb{R} f_n d\lambda \neq \int_\mathbb{R} \lim_{n \to \infty} f_n d\lambda$.

4. Find and justify the limits

(a) $\lim_{n \to \infty} \int_\mathbb{R} 1_{[0,n]}(x) \frac{\sin x}{1 + nx^2} d\lambda(x)$.

(b) $\lim_{n \to \infty} \int_\mathbb{R} 1_{[0,e^n]}(x) \frac{x}{1 + nx^2} d\lambda(x)$.

5. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence of bounded measurable functions $f_n : X \to \mathbb{R}$. Assume that $f_n$ converges uniformly to a function $f$. Show that $f_n \to f$ in $L^1$ and that $\int_X f d\mu = \lim_{n \to \infty} \int_X f_n d\mu$. 
6. Suppose that \( f \in L^1(\mathbb{R}) \) is such that \( \int_{\mathbb{R}} |xf(x)| \, d\lambda(x) < \infty \). Show that
\[
F(t) = \int_{\mathbb{R}} f(x) \cos(tx) \, d\lambda(x)
\]
is differentiable and find \( F'(t) \).

7. Let \( f \in L^1[0, 1], f \geq 0 \). Show that
\[
\sqrt{\int_{[0,1]} f \, d\lambda(x)} \geq \int_{[0,1]} \sqrt{f(x)} \, d\lambda(x)
\]

8. Let \((X, \mathcal{A}, \mu)\) be a \( \sigma \)-finite measure space. Let \( \sigma \) be the counting measure on \( \mathbb{N} \). Finally let \( Z = \mathbb{N} \times X, \mathcal{B} = \mathcal{P}(\mathbb{N}) \otimes \mathcal{A} \) and \( \eta = \sigma \otimes \mu \).

\( \text{(a)} \) Show that \( E \subset Z \) is measurable if and only if \( E_n = \{x \in X \mid (n, x) \in E\} \) is measurable for each \( n \in \mathbb{N} \) and in that case \( \eta(E) = \sum_{n=1}^{\infty} \mu(E_n) \).

\( \text{(b)} \) For \( f : Z \to \mathbb{R} \) and \( n \in \mathbb{N} \) define \( f_n : X \to \mathbb{R} \) by \( f_n(x) = f(n, x) \). Then \( f \) is measurable if and only if \( f_n \) is measurable for all \( n \).

\( \text{(c)} \) Let the notation be as in (2) and assume that \( f \) is measurable. Then \( f \in L^1(Z, \eta) \) if and only if \( f_n \in L^1(X, \mu) \) for all \( n \in \mathbb{N} \) and \( \sum_{n=1}^{\infty} \int_X |f_n(x)| \, d\mu(x) < \infty \).

9. For \(-\infty < a < b < \infty\) let \( I = [a, b] \) and let \( \mu \) be a signed measure on \( I \) (with respect to the Borel \( \sigma \)-algebra). Let \( f(x) = \mu([a, x]) \). Then \( f \) is of bounded variation.

10. For \( 1 \leq p < \infty \) let \( \ell^p \) be the space of sequences \( \{a_n\}_{n \in \mathbb{N}} \) such that \( \sum_{n=1}^{\infty} |a_n|^p < \infty \). Thus \( \ell^p = L^p(\mathbb{N}) \) with respect to the counting measure. For \( 1 \leq p \leq q < \infty \) show that \( \ell^p \subset \ell^q \) and that \( \|\{a_n\}\|_q \leq \|\{a_n\}\|_p \).