Instructions: There are three parts. You must turn in 2 problems from Part I, 2 problems from Part II, and 1 problem from Part III. All problems have equal weight. Each problem submitted must be written on a separate sheet of paper with your name and problem number at the top. Your solutions must be complete and clear. You must show that the hypotheses of well known results that you use are satisfied. Make sure they are properly used and quoted. *Unless otherwise indicated* all references to measure and integration on the real line is in the sense of Lebesgue. You have 3 hours to complete this exam. We wish you well!

Part I: Choose 2 of the following 3 problems

1. Let $(X, \mathfrak{A}, \mu)$ be a measure space. Suppose $A_n \in \mathfrak{A}$, $A_n \subset A_{n+1}$ for $n = 1, 2, \ldots$, and $A = \bigcup_{n=1}^{\infty} A_n$. Show
$$\mu A = \lim_{n\to\infty} \mu A_n.$$
2. Let $(X, \mathfrak{A})$ be a measure space. A function $f : X \to \mathbb{R}$ is **measurable** if $f^{-1}(-\infty, a) \in \mathfrak{A}$ for all real numbers $a \in \mathbb{R}$. Show that f is measurable if and only if $f^{-1}B \in \mathfrak{A}$ for all Borel sets $B \subset \mathbb{R}$.
3. Let $E \subset \mathbb{R}$ be a measurable set. Suppose $0 < c < 1$ and
$$\ell(E \cap I) \leq c\ell(I)$$
for all intervals I. Show $\ell(E) = 0$.

Part II: Choose 2 of the following 3 problems

4. Suppose μ and ν are two finite (nonnegative) measures on a measurable space $(X, \mathfrak{A})$. Show there is a nonnegative measurable function f on X such that for all $E \in \mathfrak{A}$,
$$\int_E (1 - f)\, d\mu = \int_E f\, d\nu.$$
Hint: Consider how μ and $\mu + \nu$ are related?
5. Define $f(x) = \int_{\mathbb{R}} \cos(xy) g(y)\, dy$ for $x \in \mathbb{R}$, where $g \in L^1(\mathbb{R})$. Show f is continuous on $\mathbb{R}$.
6. Let $g_n = n\chi_{[0,n^{-3}]}$.
   (a) Show that
   $$\lim_{n\to\infty} \int_0^1 f(x) g_n(x)\, dx = 0,$$
   for all $f \in L^2[0, 1]$.
   (b) Find a function $f \in L^1[0, 1]$ for which
   $$\lim_{n\to\infty} \int_0^1 f(x) g_n(x)\, dx \neq 0.$$

Part III: Choose 1 of the following 2 problems

7. Let $(X, \mathfrak{A}, \mu)$ be a measure space. Suppose f_n is a sequence of non negative integrable functions on X that converges almost everywhere to an integrable function f, and
$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$
Let $B \in \mathfrak{A}$. Show
$$\lim_{n\to\infty} \int_B f_n \, d\mu = \int_B f \, d\mu.$$
Hint: Use Fatou's lemma.

8. Let $(X, \mathfrak{A}, \mu)$ be a finite measure space. Suppose $f_n \in L^p(X)$ for $n = 1, 2, \ldots$, $f_n \to f$ a.e., $f \in L^p(x)$, and $\|f_n\|_p \to \|f\|_p$. Show $f_n \to f$ in $L^p(X)$. Hint: Use Egoroff's theorem.