Mathematics Comprehensive Examination

Core I –Analysis

January 2015

Directions: There are two parts to the exam. Solutions to at least two problems must be turned in from each part. Solutions to 5 problems are required.

Part I.

1. Let f be a Lebesgue measurable function and B be a Borel set. Show that $f^{-1}(B)$ is a Lebesgue measurable set.

2. Let f be a non-negative Lebesgue integrable function on $\mathbb{R}$. Show that the function

$$F(x) = \int_{-\infty}^{x} f$$

is continuous.

3. If f is integrable on $\mathbb{R}$ show that $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\cos nx\,dx = 0$.

4. Provide an example of a sequence $\{f_n\}_{n=1}^{\infty}$ of bounded measurable functions on $[0,1]$ for which $\lim_{n\to\infty} \int_0^1 |f_n(x)|\,dx = 0$ but does not converge pointwise anywhere on $[0,1]$.

Part II.

1. Let g be defined by $g(0) = 0$ and $g(x) = x^2(1 - \cos\frac{1}{x})$ for $x \neq 0$. Is g of bounded variation on $[-1,1]$? If it is prove why, otherwise demonstrate that it is not.

2. Suppose that for some $M > 0$ that the function f satisfies the Lipschitz condition $|f(x) - f(y)| < M|x - y|$ for x and y in $[0,10]$. Suppose further that f is absolutely continuous. Is it true that $|f'|$ is bounded? If so show why. Conversely suppose f is absolutely continuous and $|f'|$ is bounded does f satisfy a Lipschitz condition $|f(x) - f(y)| < M|x - y|$ for some number M?

3. Let $f(x,t)$ be defined and bounded on $Q = \{(x,t) : 0 \leq x \leq 1,\, 0 \leq y \leq 1\}$ and suppose:

   (a) For each t, $f(x,t)$ is Lebesgue measurable in x.
   (b) For each $(x,t) \in Q$, $\frac{\partial f}{\partial t}$ exists and is bounded on Q.

   Show

$$\frac{d}{dt}\int_0^1 f(x,t)\,dx = \int_0^1 \frac{\partial f(x,t)}{\partial t}\,dx.$$