Instructions: There are five problems on this exam. For each, you have a choice between two problems. Write the number of each problem you work out, and write your name at the top of each page you turn in. You have three hours.

Do exactly one of the following two problems.

1A. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $f : X \to \mathbb{R}$ be a real-valued μ-measurable function.

a. Prove the following statement: For each $\epsilon > 0$, there exists a set $E \in \mathcal{M}$ such that $\mu(X \setminus E) < \epsilon$ and f is bounded on E.

b. Provide an example for which there is no set E such that $\mu(X \setminus E) = 0$ and f is bounded on E.

1B. Let μ^* be an outer measure on X and $\{A_j\}_{j=1}^{\infty}$ a sequence of disjoint μ^*-measurable sets. Prove that, for each subset $E \subset X$,

$$\mu^*\left(E \cap \left(\cup_{j=1}^{\infty} A_j\right)\right) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j)\,.$$

Do exactly one of the following two problems.

2A. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that

$$\sum_{n=1}^{\infty} \|f_n\|_1 < \infty\,.$$

a. Prove that $\sum_{n=1}^{\infty} f_n(x)$ converges for almost all $x \in X$ to an integrable function.

b. Prove that

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n\,.$$

2B. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $f : X \to \mathbb{R}$ be a non-negative μ-measurable function. Suppose that, for every $n \in \mathbb{N}$,

$$\int_X (f(x))^n \, d\mu = \int_X f(x)\, d\mu < \infty\,.$$

Prove that there is a set $E \in \mathcal{M}$ such that $f = \chi_E$ almost everywhere μ.

Do exactly one of the following two problems.

3A. Let f and g be functions in $L^1(\mathbb{R})$ with respect to Lebesgue measure. Prove the following:

a. The function $H(x, y) := f(x - y)g(y)$ is in $L^1(\mathbb{R}^2)$.

b. The function

$$f * g\,(x) := \int_{\mathbb{R}} f(x - y)\, g(y)\, dy$$

is defined almost everywhere and is an integrable function on $\mathbb{R}$.

c. $\|f * g\|_1 \leq \|f\|_1 \, \|g\|_1$.

3B. Define the following functions of real variables:

$$\begin{aligned} f(x) &= x^{-1} \sin x \\ g(x, y) &= e^{-xy} \sin x \end{aligned}$$

a. Prove that $f \notin L^1\big((0, \infty)\big)$.

b. For each positive real number b, prove that $g \in L^1\big((0, b) \times (0, \infty)\big)$.

c. Prove that

$$\lim_{b \to \infty} \int_0^b f(x)\, dx = \frac{\pi}{2}$$

by working with the integral of g over $B := (0, b) \times (0, \infty)$.

Do exactly one of the following two problems.

4A. Suppose that finite (positive) measures λ, μ, and ν defined on a σ-algebra $\mathcal{M}$ of subsets of a set X have the relationship

$$\lambda \ll \mu \ll \nu \,.$$

Prove that $\lambda \ll \nu$ and that the Radon-Nikodym derivatives are related by

$$\frac{d\lambda}{d\nu} = \frac{d\lambda}{d\mu} \frac{d\mu}{d\nu} \,.$$

4B. Let $k : [0,1] \to \mathbb{R}$ denote the Cantor ternary function[1], and define a function $F : [0,1] \to \mathbb{R}$ by

$$F(x) \;=\; k(x) \;-\; \chi_{[1/2,\,1]}(x) + \int_0^x \cos(\pi y^2)\, dy\,.$$

Answer the following as explicitly as possible.

a. Find the canonical decomposition of F into the difference of increasing functions, $F = F_+ - F_-$, where F_+ and F_- are the positive and negative variations of F. Justify your result.

b. Find the total variation function $T_F(x)$ of $F(x)$.

c. Find the Lebesgue decomposition $\mu_F \;=\; \mu_s + \mu_a$ of μ_F into singular (μ_s) and absolutely continuous (μ_a) parts with respect to Lebesgue measure. Here, μ_F is the unique Borel measure on $[0,1]$ such that $\mu_F((a,b]) = F(b+) - F(a+)$ for $0 \le a \le b \le 1$ (and $\mu_F(\{0\}) = 0$).

d. What is the Radon-Nikodym derivative of μ_a with respect to Lebesgue measure?

e. What can you say about the derivative of F?

Do exactly one of the following two problems.

5A. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let f be a μ-measurable function. Prove that

$$\lim_{p\to\infty} \|f\|_p \;=\; \|f\|_\infty\;.$$

5B. Suppose that $1 \le p \le q \le \infty$ and that $p^{-1} + q^{-1} = 1$, and work with Lebesgue measure on $\mathbb{R}$ and complex-valued functions.

a. In the case that $1 \le p \le \infty$, let $f \in L^p(\mathbb{R})$ be given. Find a function $g \in L^q(\mathbb{R})$ such that

$$\|g\|_q \;=\; 1 \quad \text{and} \quad \int fg \;=\; \|f\|_p\;.$$

(Don't forget to treat $p = 1$.)

b. Provide an example of a function f in $L^\infty(\mathbb{R})$ for which *no function* $g \in L^1(\mathbb{R})$ exists such that

$$\|g\|_1 \;=\; 1 \quad \text{and} \quad \int fg \;=\; \|f\|_\infty\;.$$

[1] Recall that the Cantor ternary function $k(x)$ is increasing with total variation 1 on $[0,1]$; $k(x)$ is constant on each of the open intervals that form the complement of the Cantor ternary set; and the Cantor ternary set has Lebesgue measure 0.