Instructions: There are three parts. You must turn in 1 problems from Part I, 2 problems from Part II, and 2 problem from Part III. All problems have equal weight. Each problem submitted must be written on a separate sheet of paper with your name and problem number at the top. Your solutions must be complete and clear. You must show that the hypotheses of well known results that you use are satisfied. Make sure they are properly used and quoted. *Unless otherwise indicated* all references to measure and integration on the real line is in the sense of Lebesgue. You have 3 hours to complete this exam. We wish you well!

Part I: Choose 1 of the following 2 problems

1. Let $f : X \to \mathbb{R}^*$ be an extended real-valued measurable function on a finite measure space $(X, \mathcal{A}, \mu)$. Suppose that $f(x)$ is finite for almost all x. Prove that for each $\epsilon > 0$ there exists a set $A \in \mathcal{A}$, with $\mu(X \backslash A) < \epsilon$, such that f is bounded on A. Give an example for which it is impossible to require that $X \backslash A$ be a μ-null set.

2. Suppose $(X, \mathcal{A}, \mu)$ is a measure space. Suppose $f_n : X \to \mathbb{R}$ is a sequence of measurable functions on X. Suppose $f_n \to 0$ in measure. (Recall that this means that for each $\epsilon > 0$ there is an $N > 0$ such that $\mu\{x \mid |f_n(x)| \geq \epsilon\} < \epsilon$ for all $n > N$.) Show there is a subsequence f_{n_k} of f_n such that $f_{n_k}(x) \to 0$ as $k \to \infty$ almost everywhere. (Do not simply quote a more general theorem here. Your proof should be based on the definition of convergence in measure given above.)

Part II: Choose 2 of the following 3 problems

3. Show there are no bounded sequences $\{a_n\}$ and $\{b_n\}$ for which
$$a_n \sin(2\pi n x) + b_n \cos(2\pi n x)$$
converges to the constant function 1 almost everywhere on the unit interval $[0, 1]$.

4. Suppose p, q, and r are all greater than 1 and are related by $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Suppose $f \in L^p[0,1]$, $g \in L^q[0,1]$, Show that $fg \in L^r[0,1]$ and
$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

5. Let $0 \leq r < 1$ and define
$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^{-r} & \text{if } 0 < x \leq 1 \end{cases}.$$
   (a) State the monotone convergence theorem.
   (b) Use the monotone convergence theorem to show that f is a Lebesgue integrable function on $[0, 1]$
   (c) Compute
$$\lim_{n\to\infty} \int_0^1 \frac{n}{1 + nx^r}\, dx.$$

Part III: Choose 2 of the following 3 problems

6. Let $(X, \mathfrak{A}, \mu)$ be a measure space. Suppose f_n is a sequence of non negative integrable functions on X that converges almost everywhere to an integrable function f, and
$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$
Let $B \in \mathfrak{A}$. Show
$$\lim_{n\to\infty} \int_B f_n \, d\mu = \int_B f \, d\mu.$$

7. Let f be absolutely continuous on $[a, b]$. Let $T_a^b(f)$ be the total variation of f on $[a, b]$. Show that
$$T_a^b(f) = \int_a^b |f'| \, .$$

8. Let $(X, \mathfrak{A}, \mu)$ be a finite measure space. Suppose $f_n \in L^p(X)$ for $n = 1, 2, \ldots$, $f_n \to f$ a.e., $f \in L^p(x)$, and $\|f_n\|_p \to \|f\|_p$. Show $f_n \to f$ in $L^p(X)$.