Comprehensive/Qualifying Examination

Real Analysis

January 2018

Instructions. You must solve 2 problems from **Part I**, 2 problems from **Part II**, and 1 problem from **Part III**. All problems have equal weight. Each solution submitted must be written on a separate sheet of paper with your name and problem number at the top. Indicate on a separate sheet the problems you **omit**.

Carefully show all your steps. You may appeal to a "well known theorem," but state it precisely and show that the hypothesis is clearly satisfied. Unless otherwise indicated, all references to measure and integration are in the sense of Lebesgue.

Part I. Choose 2 of the following 3 problems.

1. Let $\{A_n\}_{n\geq 1}$ be a sequence of Lebesgue measurable subsets of $[0,1]$. Assume that 1 is a limit point of the sequence $\{m(A_n)\}$, where m denotes the Lebesgue measure on $[0,1]$. Prove that there exists a subsequence $\{A_{n_k}\}_{k\geq 1}$ such that

$$m\left(\bigcap_{k=1}^{\infty} A_{n_k}\right) > 0.$$

2. Let $E \subset \mathbb{R}$ be a measurable set with the property that

$$m(E \cap I) \leq \frac{m(I)}{2},$$

for every open interval I (m is the Lebesgue measure on $\mathbb{R}$). Prove that $m(E) = 0$.

3. Let f be a bounded measurable function on $\mathbb{R}$ for which there is a constant $C > 0$ such that

$$\forall \epsilon > 0, \quad m(\{x \in \mathbb{R} : |f(x)| > \epsilon\}) \leq C/\sqrt{\epsilon},$$

where m is the Lebesgue measure on $\mathbb{R}$. Prove that $f \in L^1(\mathbb{R})$.

Part II. Choose 2 of the following 3 problems.

4. Let m be the Lebesgue measure on $\mathbb{R}$. Let $\{f_n\}$, $\{g_n\}$, and $\{h_n\}$ be sequences of integrable functions on $\mathbb{R}$. Suppose that f, g, and h are such that

   (i) $f, h \in L^1(\mathbb{R})$,

   (ii) $\lim_n f_n(x) = f(x)$, $\lim_n g_n(x) = g(x)$, and $\lim_n h_n(x) = h(x)$, for a.e. x,

   (iii) $f_n(x) \le g_n(x) \le h_n(x)$ for a.e. x, and

   (iv) $\lim_n \int_{\mathbb{R}} f_n\, dm = \int_{\mathbb{R}} f\, dm$, and $\lim_n \int_{\mathbb{R}} h_n\, dm = \int_{\mathbb{R}} h\, dm$.

   Prove that
$$\lim_{n\to\infty} \int_{\mathbb{R}} g_n\, dm = \int_{\mathbb{R}} g\, dm$$

5. Prove that, if f is a real-valued Lebesgue integrable function on $\mathbb{R}$, then
$$\lim_{x\to 0} \int_{\mathbb{R}} |f(x+t) - f(t)| dt = 0.$$

6. Let $\mathcal{F}$ denote the class of functions $\mathbb{R}^n \times \mathbb{R}^m \ni (\mathbf{x}, \mathbf{y}) \mapsto g(\mathbf{x}, \mathbf{y}) \in \mathbb{R}$ for which

   (i) $\mathbb{R}^m \ni \mathbf{y} \mapsto g(\mathbf{x}, \mathbf{y})$ is measurable and integrable for almost all $\mathbf{x} \in \mathbb{R}^n$;

   (ii) $\mathbb{R}^n \ni \mathbf{x} \mapsto \int_{\mathbb{R}^m} g(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$ is measurable and integrable;

   (iii)
$$\int_{\mathbb{R}^n\times\mathbb{R}^m} g =: \iint_{\mathbb{R}^n\times\mathbb{R}^m} g(\mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y} = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^m} g(\mathbf{x}, \mathbf{y})\, d\mathbf{y}\right] d\mathbf{x}.$$

   Prove that if $f_k \in \mathcal{F}$ and $f_k \nearrow f \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$, then $f \in \mathcal{F}$.

Part III. Choose 1 of the following 2 problems.

7. Let f be a measurable non-negative function on the measure space (X, Σ, μ), with $0 < \mu(X) < \infty$. Let
$$\|f\|_\infty = \sup\left\{M : \mu\left(f^{-1}(M-\delta, M]\right) > 0,\ \forall\, \delta > 0\right\}.$$

   Show that
$$\lim_{n\to\infty} \left(\int_X f(x)^n d\mu(x)\right)^{\frac{1}{n}} = \|f\|_\infty.$$

8. Assume that $f : [0,1] \to \mathbb{R}$ is a monotone increasing function. Prove that the following two statements are equivalent.

   (i) f is absolutely continuous.

   (ii) For every absolutely continuous function g on $[0,1]$, and for every $x \in [0,1]$,
$$\int_0^x f(t)g'(t)\, dt + \int_0^x f'(t)g(t)\, dt = f(x)g(x) - f(0)g(0).$$