The exam has three parts. You must turn in **two** problems from each of parts I and II and **one** problem from part III. All problems have the same weight. Only five problems will be graded. Make sure to have your name clearly written on the solution sheets. You can use "well known theorems" from the lectures, but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied. $(X, \mathcal{A})$ will always stand for a measurable state and μ will denote a measure on X. If not otherwise stated no further conditions on $X, \mathcal{A}$ or μ are made. If $A \subseteq X$ then χ_A denotes the indicator functions of the set A. λ will denote the Lebesgue measure on $\mathbb{R}$ or a given subset of $\mathbb{R}$. λ^d will denote the Lebesgue measure on $\mathbb{R}^d$ or a subset thereof. $\overline{\mathbb{R}}$ denotes the set of extended real numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$. The convolution of two measurable functions on $\mathbb{R}^d$ is defined by $f * g(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\, d\lambda(y)$ whenever $y \mapsto f(y)g(x-y)$ is integrable.

PART I

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1) a) Let $f : X \to \overline{\mathbb{R}}$ be measurable and such that f is not almost everywhere infinite. Show that there exists a subset $S \subseteq X$ such that $\mu(S) > 0$ and f is bounded on S.

b) Assume that $f \in \mathcal{L}^1(X, \mu)$. Show that $\{x \in X \mid f(x) \neq 0\}$ is σ-finite.

2) Let $E \subseteq \mathbb{R}$ be measurable set with the property that $\lambda(E \cap I) \leq \lambda(I)/2$ for every open interval I. Show that $\lambda(E) = 0$.

3) Assume that $f \in \mathcal{L}^1(\mathbb{R}^d, \lambda^d)$. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$. Show that $f * \varphi$ is continuously differentiable.

4) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of measurable sets of finite measure and assume that the sequence $f_n = \chi_{A_n}$ converges in the L^1-norm to a function $f \in L^1(X, \mu)$. Show that there exists a measurable set A of finite measure such that $f = \chi_A$ a.e.

PART II

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5) a) Assume that (f_n), $f_n : X \to \mathbb{R}$, is a sequence of measurable functions. Show that the set $S = \{x \in X \mid \lim\limits_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable.

b) Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^1(X)$ converging in the L^1-norm to $f \in \mathcal{L}^1(X)$. Show that $f_n \to f$ in measure.

6) For $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \chi_{[-n,n]}(x) \sin\left(\frac{\pi x}{n}\right)$.

a) Determine $f(x) = \lim\limits_{n\to\infty} f_n(x)$ and show that the sequence (f_n) converges uniformly on compact subsets of $\mathbb{R}$. Does the sequence converges uniformly on $\mathbb{R}$?

b) Show that $\int_{\mathbb{R}} f\,d\lambda = \lim_{n\to\infty} \int_{\mathbb{R}} f_n d\lambda$ but there is no positive integrable function g such that $|f_n(x)| \leq g(x)$ for almost all $x \in \mathbb{R}$.

7) For each of the following two problems problems (a) and (b), check whether the limit exists. If so, find its value.

a) $\lim_{n\to\infty} \int_{[1,n)} \left(1 - \frac{x}{n}\right)^n d\lambda(x)$.

b) $\lim_{n\to\infty} \int_{[1,2n)} \left(1 - \frac{x}{n}\right)^n d\lambda(x)$.

8) Let $I = [a,b] \subset \mathbb{R}$ be a compact interval. Assume that $f : I \to \mathbb{R}$ is continuous. Show that f is integrable and that $\int_I f(x)\,d\lambda(x) = \int_a^b f(x)\,dx$ (the Lebesgue integral is the same as the Riemann integral).

9) a) Assume that $\mu(X) < \infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^1(X,\mu)$ that converges uniformly on X to the function f. Show that f is integrable and

$$\lim_{n\to\infty} \int_X f_n\,d\mu = \int_X f\,d\mu\,.$$

b) Give an example showing that the conclusion in (a) is not always valid if $\mu(X) = \infty$.

PART III

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10) Let $g : \mathbb{R} \to \mathbb{R}$ be measurable, $g \geq 0$, $g(x) = 0$ for $|x| > 1$ and $\int_{\mathbb{R}} g\,d\lambda = 1$. Let $g_n(x) = ng(nx)$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous then fg_n is integrable and $\lim_{n\to\infty} \int_{\mathbb{R}} fg_n d\lambda = f(0)$.

11) a) Assume that $f : [0,\infty) \to \mathbb{R}$ is monotone (increasing or decreasing). Show that f is measurable.

b) Let ν be a Borel measure on $[0,\infty)$ such that $\varphi(t) = \nu([0,t))$ is finite for all $t \geq 0$. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $f \in M(X)_+$. Show that

$$\int_X \varphi(f(x))\,d\mu(x) = \int_{[0,\infty)} \mu(\{x \in X \mid f(x) > t\})\,d\nu(t)\,.$$