Instructions: You must solve 2 problems from Part I, 2 problems from Part II, and 1 problem from Part III. All problems have equal weight. Each solution submitted must be written on a separate sheet of paper with your name and problem number at the top. Indicate on a separate sheet the problems you omit.

Carefully show all your steps. You may appeal to a well known theorem, but state it precisely and show that the hypothesis is clearly satisfied. Unless otherwise indicated, all references to measure and integration are in the sense of Lebesgue, and m denotes Lebesgue measure on $\mathbb{R}$. Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

Part I. Choose 2 of the following 3 problems.

1. If E is a measurable subset of $[0,1]$, prove that there is a measurable subset $A \subset E$ such that $m(A) = \frac{1}{2}m(E)$.
2. Let $Q = [0,1]^d$ be the unit cube in $\mathbb{R}^d$ and suppose $f : Q \to \mathbb{R}$ is a continuous function. Show that the Lebesgue measure of the compact set
$$\Gamma = \{(x, f(x)) \,|\, x \in Q\}$$
in $\mathbb{R}^{d+1}$ is zero. (Hint: Use uniform continuity.)
3. Prove that every non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ is Borel-measurable.

Part II. Choose 2 of the following 3 problems.

1. Let f be a non-negative integrable function on $[0,1]$. Suppose that for every $n \in \mathbb{N}$,
$$\int_{[0,1]} (f(x))^n dm = \int_{[0,1]} f(x) dm.$$
Show that f is almost everywhere the indicator (that is, characteristic) function for some measurable set.
2. Find the limit and justify your answer:
$$\lim_{n\to\infty} \int_{(a,\infty)} \frac{n}{1+n^2x^2} dm.$$
(Hint: The answer depends on whether $a > 0, = 0$, or < 0.)
3. Let $f \in L^1(0,\infty)$. Suppose that $\int_{(0,\infty)} x|f(x)| dm < \infty$. Prove that the function
$$g(y) = \int_{(0,\infty)} e^{-xy} f(x)\, m(dx)$$
is differentiable at every $y \in (0,\infty)$.

Part III. Choose 1 of the following 2 problems.

1. If f is continuous on the interval $[a, b]$ and has a bounded derivative in (a, b), show that f is of bounded variation on $[a, b]$. Is boundedness of f' necessary for f to be of bounded variation? Justify your answer.

2. Suppose that ν is a signed measure and μ is a measure on a measurable space $(X, \mathcal{F})$. Show that the following are equivalent:

   (i) $\nu << \mu$

   (ii) $|\nu| << \mu$

   (iii) $\nu^+ << \mu$ and $\nu^- << \mu$ where $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν.