The exam has three parts. You must turn in **two** problems from each of parts I and II and **one** problem from part III. All problems have the same weight. Only five problems will be graded. Make sure to have your name clearly written on the solution sheets. You can use "well known theorems" from the lectures, but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied.
$(X, \mathcal{A})$ will always stand for a measurable space and μ will denote a positive measure on X. If not otherwise stated no further conditions on $X, \mathcal{A}$ or μ are made. The σ-algebra on $\mathbb{R}^d$, or a subset of $\mathbb{R}^d$, is the Borel σ-algebra and λ^d will denote the Lebesgue measure. If $d = 1$ we write $\lambda = \lambda^1$. $\overline{\mathbb{R}}$ denotes the set of extended real numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$. $C^1(\mathbb{R}^n)$ denotes the space of once continuously differentiable functions (g' exists and is continuous) and $C_c^1(\mathbb{R})$ denotes the subspace of $g \in C^1(\mathbb{R})$ such that g has compact support.

PART I

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1) Let $f : X \to \overline{\mathbb{R}}$. Then f is measurable if and only if $f^{-1}((-\infty, q]) \in \mathcal{A}$ for all $q \in \mathbb{Q}$.

2) Let $\{E_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of sets in $\mathcal{A}$ ($E_{n+1} \subseteq E_n$ for all $n \in \mathbb{N}$). Let $E = \bigcap_n E_n$.

(i) Show that if $\mu(X) < \infty$ then $\mu(E) = \lim_{n\to\infty} \mu(E_n)$.

(ii) Let $f \in L^1(X)$, $f \geq 0$. Show that $\lim_{n\to\infty} \int_{E_n} f\, d\mu = \int_E f\, d\mu$.

3) a) Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^1(X)$ converging in the L^1-norm to a function $f \in \mathcal{L}^1(X)$. Show that $f_n \to f$ in measure.

b) Show by an example that the conclusion in (a) does not hold if $\mu(X) = \infty$.

4) Let $Q = [0, 1]^d \subset \mathbb{R}^d$ and suppose that $f : Q \to \mathbb{R}$ is continuous. Show that

$$A = \{(x, f(x)) \mid x \in Q\}$$

is measurable and $\lambda^{d+1}(A) = 0$.

PART II

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5) Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions $f_n : X \to \mathbb{R}$. Show that the set $S = \{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable.

6) For $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \chi_{[-n,n]}(x) \cos\left(\frac{\pi x}{n}\right)$.

a) Determine $f(x) = \lim_{n\to\infty} f_n(x)$ and show that the sequence $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{R}$. Does the sequence converges uniformly on $\mathbb{R}$?

b) Show that $\lim_{n\to\infty} \int_{\mathbb{R}} f_n\, d\lambda$ exists but that $\lim_{n\to\infty} f_n$ is not integrable.

7) Let $\mathrm{I} = (e, \infty)$ and let $f_n(x) = \dfrac{n}{x(\log x)^n}$ for $x \in \mathrm{I}$ and $n \in \mathbb{N}$.

(i) For which $n \in \mathbb{N}$ is f_n integrable on I.

(ii) Determine $\lim_{n\to\infty} f_n(x)$, $x \in \mathrm{I}$, and $\lim_{n\to\infty} \int_{\mathrm{I}} f_n(x)\, d\lambda(x)$.

(iii) Does the sequence $\{f_n\}_n$ satisfy the assumptions of Lebesgue's dominated convergence theorem?

8) Let $g \in C_c^1(\mathbb{R})$ and let $f \in L^1(\mathbb{R})$. Show that the function

$$F(t) = \int_{\mathbb{R}} g(tx) f(x)\, d\lambda(x)$$

is continuously differentiable and that

$$F'(t) = \int_{\mathbb{R}} g'(tx) x f(x)\, d\lambda(x).$$

9) a) Assume that $\mu(X) < \infty$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^1(X,\mu)$ that converges uniformly on X to the function f. Show that f is integrable and

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu\,.$$

b) Give an example showing that the conclusion in (a) is not always valid if $\mu(X) = \infty$.

PART III

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10) (a) Let $f, g \in L^1(\mathbb{R}^d)$. Show that the function $y \mapsto f(y)g(x-y)$ integrable for almost all $x \in \mathbb{R}^d$ and that $h(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\, d\lambda^d(y)$ is an integrable function on $\mathbb{R}^d$.

(b) Assume that $d = 1$ and that $f \in L^1(\mathbb{R})$ and $g \in C_c^1(\mathbb{R}^n)$. Then $f * g \in C^1(\mathbb{R})$ and $(f*g)'(x) = f * g'(x)$.

11) Let $\mathcal{H}$ be a Banach space. Let $\{T_n\}_{n\in\mathbb{N}}$ and $\{S_n\}_{n\in\mathbb{N}}$ be sequences of bounded linear operators $T_n, S_n : \mathcal{H} \to \mathcal{H}$. Assume that there exits bounded linear operators $T, S : \mathcal{H} \to \mathcal{H}$ such that for all $v \in \mathcal{H}$ we have

$$\|T_n v - Tv\|, \|S_n v - Sv\| \to 0 \quad \text{as } n \to \infty.$$

Show the following:

(i) If $\{v_n\}_n$ is a sequence in $\mathcal{H}$ converging to $v \in \mathcal{H}$. Then

$$\lim_{n\to\infty} \|T_n v_n - Tv\| = 0.$$

(ii) If $v \in \mathcal{H}$ then

$$\lim_{n\to\infty} \|T_n S_n v - TSv\| = 0.$$

12) Assume that μ, ν are two positive measure on $(X, \mathcal{A})$ and that $\mu \ll \nu$. Denote by $g = \dfrac{d\nu}{d\mu}$ the Radon-Nykodym derivative.

(i) Assume that $f \in L^1(X, \nu)$. Show that $fg \in L^1(X, \mu)$ and

$$\int_X f\, d\nu = \int_X fg\, d\mu.$$

(ii) Recall that μ and ν are equivalent if and only if $\mu \ll \nu \ll \mu$. Show that μ and ν are equivalent if and only if $g \neq 0$ a.e.

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