Introduction to Applied Mathematics
Qualifying Exam, August 2025

The exam is 3 hours. Each problem is worth 20 points, and you can do a maximum of five problems for a total potential score of 100 points. Do at least one problem from each category.

Recall that $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space.

Category I: Continuum Mechanics

1. (20 points)
Let a deformation map $\varphi : \mathbb{R}^{3+1} \to \mathbb{R}^3$ define a time-varying spatial coordinate $x = \varphi(X,t)$ and spatial velocity field $v(x,t) = \partial_t \varphi(X,t)$; and define $F(X,t) = \nabla^X \varphi(X,t)$.

a. Prove
$$\frac{\partial}{\partial t} \det F(X,t) = \det F(X,t)(\nabla^x \cdot v)(x,t)\bigg|_{x=\varphi(X,t)}.$$

b. Prove
$$\frac{d}{dt} v(x,t) = \frac{\partial}{\partial t} v(x,t) + (\nabla^x v(x,t))\, v(x,t)$$
where $\frac{d}{dt}$ is understood as the partial time derivative of *fields in material coordinates* (X,t).

2. (20 points)
Retain the notation from Problem 1. Define an evolving body in $\mathbb{R}^3$ by starting with a set $B_0 \subset \mathbb{R}^3$ and putting $B_t = \varphi(B_0, t)$. The body at time t has density field $\rho(x,t)$, Cauchy stress tensor $S(x,t)$, and body force $\rho(x,t)b(x,t)$. Assume all fields are smooth. Let the kinetic energy of $\Omega_t \subset B_t$ be defined as
$$K[\Omega_t] = \int_{\Omega_t} \frac{1}{2} \rho(x,t)\, v(x,t) \cdot v(x,t) dV_x.$$
Consider the following balance equations in spatial coordinates
$$\rho \frac{d}{dt} v = \nabla^x \cdot S + \rho b, \quad \partial_t \rho + \nabla^x \cdot (\rho v) = 0, \quad S = S^T.$$
Here $\frac{d}{dt}$ is understood as the partial derivative in time *in material coordinates*.

a. Prove
$$\nabla^x \cdot (S^T v) = (\nabla^x \cdot S) \cdot v + S : \nabla^x v.$$

b. Prove that for any $\Omega_t = \varphi_t(\Omega)$, $\Omega \subset B$ open, then
$$\int_{\Omega_t} \rho v \cdot \frac{d}{dt} v \, dV_x + \int_{\Omega_t} S : \mathrm{sym}(\nabla^x v)\, dV_x = \int_{\partial\Omega_t} v \cdot Sn \, dA_x + \int_{\Omega_t} \rho b \cdot v \, dV_x$$

3. (20 points)
Consider an invertible smooth map

$$\varphi : (0,1)^3 \to W \subset \mathbb{R}^3$$

satisfying $\det \nabla^X \varphi(X) > 0$. Consider the set of integers $\mathbb{Z}/N = \{1, 2, \cdots N\}$. Let $G_N = (\mathbb{Z}/N)^3$ and define

$$P_N = \{(\frac{i_1}{N}, \frac{i_2}{N}, \frac{i_3}{N}) : (i_1, i_2, i_3) \in G_N\},$$

a discretization of the cube. This can be considered a *reference configuration.* Suppose a point charge is placed at each point with charge $q_N = \frac{Q}{N^3}$ for Q constant. Then we define a distribution $\rho_N \in \mathcal{D}'(W)$ corresponding to charge density given by

$$\rho_N := \sum_{p \in P_N} q_N \, \delta_{\varphi(p)}$$

where $\langle \delta_x, \phi \rangle := \phi(x)$ for $\phi \in C_c^\infty(W)$. Show $\rho_N \to \langle \rho, \cdot \rangle \in \mathcal{D}'$ for some $\rho \in C(W)$. Find ρ.

Category II: Fourier Analysis

4. (20 points) Consider the PDE $-\triangle u + \gamma u = f$ for $f \in \mathcal{S}(\mathbb{R}^3)$ and $\gamma > 0$ a scalar.

a. Compute the solution $u(x)$ using Fourier analysis *(Show your derivation, but you do not need to prove it is a solution in this part).*

b. Prove $u \in \mathcal{S}(\mathbb{R}^d)$, and conclude it is a classical solution to the PDE.

5. (20 points) Suppose $f, g \in C^\infty_{\text{per}}([0,1]^d)$ with a Fourier series $f(x) = \sum_{n \in \mathbb{Z}^d} f_n e^{2\pi i n x}$ and $g(x) = \sum_{n \in \mathbb{Z}^d} g_n e^{2\pi i n \cdot x}$.

a. Derive a formula for the Fourier coefficients of $f(x)g(x)$ in terms of the g_n's and f_n's.

b. Derive a formula for the Fourier coefficients of $-\nabla \cdot (A \nabla f)$ where $A \in \mathbb{C}^{d \times d}$.

6. (20 points)
Suppose $V(x) = \sum_{n \in \mathbb{Z}^d} V_n e^{2\pi i n \cdot x}$ is a smooth periodic vector-valued function. Suppose $u \in \mathcal{S}(\mathbb{R}^d)$. Find in terms of $\hat{u}$ and V_n's a formula for the Fourier transform of $f(x) = \langle V(x), \nabla u(x) \rangle$ where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product over $\mathbb{C}^d$.

Category III: Weak-form PDEs & Distribution Theory

7. **(20 points)** Let $\Omega \subset \mathbb{R}^3$ be a bounded set with C^1 boundary, and recall that $\triangle = \nabla \cdot \nabla$.

a. Suppose $u, v \in C^2(\mathbb{R}^3)$. Show

$$\int_{\Omega} (u \triangle v - v \triangle u) dV = \int_{\partial\Omega} \Big(u \nabla v - v \nabla u \Big) \cdot n \, dS$$

where n is the outward pointing normal to the surface $\partial\Omega$.

b. For $y \in \Omega$ define $G_y(x) = \frac{1}{4\pi|x-y|}$, and note that $-\triangle G_y = \delta_y$ in the sense of distributions. For $\phi \in C_c^\infty(\Omega)$ a test function, δ_y is defined by $\langle \delta_y, \phi \rangle = \phi(y)$. If $v \in C^2(\mathbb{R}^3)$ satisfying $\triangle v = 0$, show

$$v(y) = \int_{\partial\Omega} (G_y(x) \nabla v(x) - v(x) \nabla G_y(x)) \cdot n \, dS_x.$$

8. **(20 points)**
For $y \in \mathbb{R}$, let δ_y be the distribution defined by $\langle \delta_y, \phi \rangle = \phi(y)$ where $\phi \in C_c^\infty(\mathbb{R})$, i.e. smooth and compactly supported. Consider the distribution

$$F_h = h^{-2}(\delta_h - 2\delta_0 + \delta_{-h}).$$

Find a limiting distribution F such that $F_h \to F$ in the sense of distributions.

9. **(20 points)** Consider the operator $(-\triangle + 1)^{-1} : L^2([0,1]) \to L^2([0,1])$. Show it is a compact operator, *i.e.* can be approximated in operator norm to any desired accuracy by an operator of finite rank.