

There are twelve problems below in three parts.

Complete FIVE problems, doing at least ONE problem from each part.

All problems have the same weight of twenty points. Please mark the problems you want to be graded, and make sure to have your name clearly written on the solution sheets. You can use “well known theorems” from lectures, the theory of metric/Hilbert spaces, but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied, otherwise you might not get full credit.

Part A (Continuum Mechanics):

(1) Let $\mathcal{V} = \mathbb{R}^3$. Consider the linear elastic equation $\rho_0 \partial_{tt}^2 u = \nabla \cdot [A(\nabla u)]$ for $A \in \mathcal{V}^4$, u a smooth vector field, and $\rho_0 > 0$ a constant. Suppose, for $\lambda, \mu > 0$ and any $H \in \mathcal{V}^2$, that A takes the form

$$A(H) = \lambda(\text{tr} H)I + 2\mu \text{Sym}(H).$$

a) Derive the Navier equation $\rho_0 \partial_{tt}^2 u = \mu \Delta u + (\lambda + \mu) \nabla(\nabla \cdot u)$.

b) Assume the PDE is defined over $\mathbb{R}^3$. Derive the solution for $u(t, x)$ using Fourier analysis (no need to prove it is a classical solution, you’re only asked to calculate). Assume $u(0, x) = f(x)$ and $\partial_t u(0, x) = g(x)$ for f and g smooth and compactly supported vector fields.

(2) Define a body B_t with density field $\rho(t, x)$, Cauchy stress tensor $S(t, x)$ and body force $\rho(t, x)b(t, x)$ written as spatial fields. Assume all fields are smooth. Consider the kinetic energy of $\Omega_t \subset B_t$ defined as $K[\Omega_t] = \int_{\Omega_t} \frac{1}{2} \rho(t, x) v(t, x) \cdot v(t, x) dV_x$. Consider the following balance equations in spatial coordinates ($\frac{d}{dt}$ is the material derivative): $\rho \frac{d}{dt} v = \nabla^x \cdot S + \rho b$, $\partial_t \rho + \nabla^x \cdot (\rho v) = 0$, $S = S^T$.

a) Prove $\nabla^x \cdot (S^T v) = (\nabla^x \cdot S) \cdot v + S : \nabla^x v$.

b) Prove that for any $\Omega_t = \phi_t(\Omega)$, $\Omega \subset B_0$ open, then

$$\int_{\Omega_t} \rho v \cdot \frac{d}{dt} v dV_x + \int_{\Omega_t} S : \text{Sym}(\nabla^x v) dV_x = \int_{\partial \Omega_t} v \cdot S \hat{n} dA_x + \int_{\Omega_t} \rho b \cdot v dV_x.$$

(3) For $A \in \mathcal{V}^2$ a 2-tensor, define $f(A) = A^{-1}$ (defined only on the open subset of $\mathcal{V}^2$ of invertible 2-tensors). For an arbitrary $B \in \mathcal{V}^2$, show that $Df(A)B = -A^{-1}BA^{-1}$. HINT: use that $Af(A) = I$.

(4) A one-form $F(\vec{x})$ on $(0, \infty)^3$ and a change of variables $\vec{x} = \Phi(\vec{y})$ are given below. Find the associated one-form $\tilde{F}(\vec{y})$ in the new variables.

$$F(x_1, x_2, x_3) = \begin{pmatrix} x_2^2 \\ x_3/x_2 \\ x_1 \end{pmatrix} \quad \Phi(y_1, y_2, y_3) = \begin{pmatrix} 4y_2 - y_1 \\ 2y_3 \\ y_1 y_3 \end{pmatrix}.$$

Part B (Fourier Analysis):

- (1) Consider the inhomogeneous Helmholtz equation $(1 - \Delta)u(x) = f(x)$ for $x \in \mathbb{R}^d$.
 a) Suppose $f \in \mathcal{S}(\mathbb{R}^d)$. Find $u(x)$ that solves the PDE such that $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$.
 b) Suppose $f_n \in \mathcal{S}(\mathbb{R}^d)$ and $f_n \rightarrow f$ in L^2 -norm. If u_n solves $(1 - \Delta)u_n(x) = f_n(x)$, prove that $u_n \rightarrow u$ in L^2 -norm.

- (2) Consider $\sigma : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ smooth. Define the operator Q acting on Schwartz functions by

$$Q\psi(x) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma(x, \xi) \psi(y) e^{2\pi i(x-y) \cdot \xi} dy d\xi.$$

- a) Show that, if $\sigma(x, \xi) = q(x) + \xi^T A \xi$ for A a fixed $d \times d$ positive definite matrix, that Q can be written as a linear differential operator. Find that linear differential operator.
 b) Suppose $Q\psi(x) = g(x) \cdot \nabla \psi(x)$ for some vector-valued continuous and bounded function $g(x)$. Find $\sigma(x, \xi)$ for such an operator.
- (3) Two functions in $L^2([-1/2, 1/2])$ have Fourier series $f(x) = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{2\pi i k x}$, $g(x) = \sum_{m \in \mathbb{Z}} \hat{g}_m e^{2\pi i m x}$.
 a) Give a formula for the Fourier series coefficients for the product $f(x)g(x)$.
 b) If $\hat{f}_k = 0$ whenever $|k| < 20$ and $\hat{g}_m = 0$ whenever $|m| \geq 10$, prove that

$$\int_{-1/2}^{1/2} f(x)g(x) \sin(2\pi x) dx = 0.$$

- (4) Find a vector-valued function $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ solving $-\Delta u + \nabla(\nabla \cdot u) + u = f$ where each component of f is in Schwartz class. Prove that each component of u is also Schwartz.

Part C (Distributions and Operator Theory):

- (1) This problem shows that multiplication of distributions is not a continuous operation (even when it is defined). Prove that, in $\mathcal{D}'(\mathbb{R})$, $\lim_{n \rightarrow \infty} \sin(nx) = 0$ but $\lim_{n \rightarrow \infty} \sin^2(nx) \neq 0$.

- (2) Let $\Omega \subset \mathbb{R}^3$ be bounded and open and $0 < \tau_1 < \tau(x) < \tau_2$ be measurable. Consider, in the weak sense, the PDE $\nabla \cdot A(x) \nabla u(x) + \lambda \tau(x) u(x) = f(x)$ for $f \in L^2(\Omega)$ where $A(x)$ is a 3×3 self-adjoint, positive definite, and bounded matrix. Show there exists a weak solution for all λ except possibly a countable set in $H_0^1(\Omega)$, which is the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\psi\|_1^2 := \int_\Omega \nabla \psi(x) \cdot A(x) \nabla \psi(x) dx$.

- (3) On the Hilbert space $H_0^1([-1, 1])$ with inner product given by $\langle f, g \rangle = \int_{-1}^1 f'(x) \overline{g'(x)} dx$, define the linear functional $\ell(f) = f(0)$. NOTE: all elements of $H_0^1([-1, 1])$ are continuous.

- a) Prove that ℓ is a bounded linear functional.
 b) Find an element $\psi \in H_0^1([-1, 1])$ such that $\ell(f) = \langle f, \psi \rangle$.

- (4) On $L^2([-1/2, 1/2])$, define the linear operator $\mathcal{R}$ in terms of the Fourier series as follows: $\mathcal{F}(\mathcal{R}u)_n = \hat{u}_n / (1 + |n|)$. Prove that $\mathcal{R}$ is a compact operator.