**Qualifier Exam** **Applied Analysis** **January 2026**

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**There are twelve problems below in three parts.**
**Complete <u>FIVE</u> problems, doing at least <u>ONE</u> problem from each part.**
All problems have the same weight of twenty points. Please mark the problems you want to be graded, and make sure to have your name clearly written on the solution sheets. You can use "well known theorems" from lectures, the theory of metric/Hilbert spaces, but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied, otherwise you might not get full credit.

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**Part A (Continuum Mechanics):**

**(1)** Suppose $\vec{v} \in C^\infty(\mathbb{R}^3; \mathcal{V})$ is such that $\vec{v}(x_1, x_2, x_3, t) = \nabla^\perp \psi(x_1, x_2, t) = \partial_2 \psi \hat{e}_1 - \partial_1 \psi \hat{e}_2$ for some scalar function $\psi \in C^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$.

**a)** Show that there is a scalar field $f(x_1, x_2, t)$ such that $\nabla \times \vec{v} = f \hat{e}_3$ and $f = -\Delta \psi$.

**b)** If, for some smooth pressure field $p$, we have that $\vec{v}$ satisfies the Navier-Stokes equation

$$\rho_0 \left( \partial_t v_j + v_i \partial_i v_j \right) = \mu \Delta v_j - \partial_j p,$$

prove that $\partial_t f + \vec{v} \cdot \nabla f = \frac{\mu}{\rho_0} \Delta f$. Here $\rho_0$ and $\mu$ are positive constants.

**(2)** Consider the tensor $S = -y \hat{e}_1 \otimes \hat{e}_2 - y \hat{e}_2 \otimes \hat{e}_1 + x \hat{e}_1 \otimes \hat{e}_3 + x \hat{e}_3 \otimes \hat{e}_1 + 2 \hat{e}_3 \otimes \hat{e}_3$ on $\mathbb{R}^3$.

**a)** Compute the characteristic polynomial for $S$.

**b)** Compute the eigenvalues of $S$ for $(x, y, z)$ on the line $\{x = 0, \quad y = 1\}$.

**c)** Compute $\nabla \cdot S$.

**(3)** A fluid body in the upper half plane $\{(x, y) : y > 0\}$ satisfies the three equations that conserve mass, momentum, and angular momentum:

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0, \qquad \rho \left( \partial_t \vec{v} + (\nabla \vec{v}) \vec{v} \right) = \nabla \cdot S, \qquad S = S^T$$

**a)** If we explicitly have $\vec{v} = (1 + x^2) \hat{e}_1 + 2xy \hat{e}_2$, and if $\rho$ depends only on $y$, determine $\rho$.

**b)** If, with the $\vec{v}$ and $\rho$ above, we also knew that $S_{11} = 0$ and $\lim_{y \to \infty} S_{12} = 0$, determine $S$.

**(4)** Consider an invertible smooth map $\phi : (0,1)^3 \to W \subset \mathbb{R}^3$ satisfying $\det \nabla^X \phi(X) > 0$. Consider the set of integers $\mathbb{Z}/N = \{1, 2, \cdots, N\}$. Let $G_N = (\mathbb{Z}/N)^3$ and define

$$P_N = \left\{ \left( \frac{i_1}{N}, \frac{i_2}{N}, \frac{i_3}{N} \right) : (i_1, i_2, i_3) \in G_N \right\},$$

discretizing the cube. Suppose a point charge is placed at each point with charge $q_N = Q/N^3$ for $Q > 0$ a constant. We define the distribution $\rho_N \in \mathcal{D}'(W)$ corresponding to charge density given by

$$\rho_N := \sum_{p \in P_N} q_N \delta_{\phi(p)}.$$

Show that, for some function $\rho \in C(W)$, $\rho_N \to \rho$ as distributions. Find $\rho$.

**Part B (Fourier Analysis):**

**(1)** Let $g \in C(\mathbb{R})$ be periodic with period 1. Let $\alpha \in \mathbb{R}$ be irrational. Define the sequence

$$G_N = \frac{1}{2N+1} \sum_{n=-N}^{N} g(\alpha n).$$

Prove that $G_N \to \int_{-1/2}^{1/2} g(x)dx$ as $N \to \infty$. HINT: you may assume that, for such $g$, the Fourier sum converges uniformly.

**(2)** Consider the inhomogeneous Helmholtz equation $(1-\Delta)u(x) = f(x)$ for $x \in \mathbb{R}^d$.

**a)** Suppose $f \in \mathcal{S}(\mathbb{R}^d)$. Find $u(x)$ that solves the PDE such that $u(x) \to 0$ as $|x| \to \infty$.

**b)** Suppose $f_n \in \mathcal{S}(\mathbb{R}^d)$ and $f_n \to f$ in the sense of $L^2$. If each $u_n$ solves $(1-\Delta)u_n(x) = f_n(x)$, show that $\{u_n\}_n$ converges in $L^2$ to some limiting function $u$ such that, for every test function $\psi \in \mathcal{S}(\mathbb{R}^d)$, we have $\langle (1-\Delta)\psi, u\rangle = \langle \psi, f\rangle$.

**(3)** Consider the function $f(x) = e^{-|x|}$ on $\mathbb{R}$.

**a)** Find the Fourier transform of $f$. HINT: $\int_a^b \exp(\gamma x)dx = \frac{1}{\gamma}\left(\exp(\gamma b) - \exp(\gamma a)\right)$ for any $\gamma \in \mathbb{C}$.

**b)** Which $H^s$ spaces does $f$ belong to? (prove your claim).

**(4)** Define the space of functions $H^s(\mathbb{R})$ and prove that any $f \in H^1(\mathbb{R})$ is $\frac{1}{2}$-Hölder continuous. That is, there exists $M \geq 0$ such that for all $x, y \in \mathbb{R}$ with $|x-y| \leq 1$, we have

$$|f(x) - f(y)| \leq M|x-y|^{1/2}.$$

**Part C (Distributions and Operator Theory):**

**(1)** Let $\Omega \subset \mathbb{R}^3$ be bounded and open. Consider the following PDE in the weak sense:

$$\nabla \cdot A(x)\nabla u(x) + \lambda u(x) = f(x)$$

for $f \in L^2(\Omega)$ where $A(x) \in \mathbb{R}^{3\times 3}$ is self-adjoint, positive-definite, and bounded. Show there exists a weak solution $u \in H^1_{A,0}(\Omega)$, which is the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\psi\|^2 := \int_\Omega \nabla\psi(x) \cdot A(x)\overline{\nabla\psi(x)}dx$, for all $\lambda$ (with at most countably many exceptions).

**(2)** Recall the delta distribution $\delta$ has $\langle \delta, \phi\rangle = \phi(0)$ for all test functions $\phi$.

**a)** Give the definition of tempered distributions.

**b)** Determine the Fourier transform of the delta-distribution: $\mathcal{F}\delta$.

**c)** Find $\mathcal{F}(\mathcal{F}\delta)$.

**(3)** Say $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator that is *compact* and *bijective* (so $T^{-1}$ also exists and is linear). Prove, however, that $T^{-1}$ must be unbounded (we assume that $\dim(\mathcal{H}) = \infty$).

**(4)** On $H^1_0([0,1])$ with the norm $\|f\|_{H^1} = \|\frac{d}{dx}f\|_{L^2}$, consider the inclusion operator $\mathcal{I} : H^1_0([0,1]) \to L^2([0,1])$ defined by $\mathcal{I}f = f$. Show that $\mathcal{I}$ is a compact operator.