Introduction to Applied Mathematics
Qualifying Exam, January 2024

The exam is 3 hours. Each problem is worth 20 points, and you can do a maximum of five problems for a total potential score of 100 points. Pick at least one from each category.

Category I: Continuum Mechanics

1. Suppose we had a system with a fixed uniform rate of mass decay described as follows. Let \( x = \varphi(X, t) = \varphi_t(X) \) be the map from material coordinates to spatial coordinates, and let \( \rho(x, t) \) be the spatial mass density field and let \( v(x, t) = \frac{\partial}{\partial t} \varphi(X, t) \). Assume all fields are smooth. Suppose for \( \Omega \subset B \) open where \( B \) is the body and \( \Omega_t = \varphi_t(\Omega) \) that we have the mass function

\[
\text{mass}[\Omega_t] = \int_{\Omega_t} \rho(x, t)dV_x.
\]

Suppose

\[
\frac{d}{dt} \text{mass}[\Omega_t] = -\int_{\Omega_t} \gamma(x, t)\rho(x, t)dV_x \tag{1}
\]

for some smooth function \( \gamma(x, t) > 0 \). From the mass relation in (1), derive the PDE in spatial coordinates

\[
\frac{\partial}{\partial t} \rho(x, t) + \nabla \cdot (\rho(x, t)v(x, t)) = -\gamma(x, t)\rho(x, t), \quad x \in \varphi_t(B), \; t \geq 0. \tag{2}
\]

2. Consider the linear balance law in spatial coordinates for constant density

\[
\rho_0 \frac{d}{dt} v = \nabla \cdot S + \rho_0 b
\]

for body force field \( b \), Cauchy stress tensor \( S \), density \( \rho_0 \neq 0 \), and velocity \( v \). \( \frac{d}{dt} \) is defined as the partial derivative in time for fields in material coordinates. Assume all fields are smooth. Assume there is a smooth scalar pressure field \( p(x, t) \) and \( I \) is the identity matrix such that

\[
S = -pI + \mu(\nabla^2 v + \nabla v^T),
\]

and assume \( \nabla^2 \cdot v = 0 \), the incompressibility condition.

(a) (10 points) Derive the equation:

\[
\rho_0[\partial_t v + (\nabla^2 v)v] = \mu\Delta^2 v - \nabla^2 p + \rho_0 b.
\]
(b) **(10 points)** Suppose \( v(x, t) = (0, 0, v_3(x_1, x_2, t))^T, b = 0, \) and \( p(x) = p_0 > 0 \) for \( x = (x_1, x_2, x_3). \) For fixed \( x_3, \) derive a scalar-valued PDE in \( x_1, x_2 \) describing the dynamics of the fields \( v \) and \( p. \)

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3. Consider the electric D field arising from a single charge living at \( y \in \mathbb{R}^3 \) defined by

\[
D_y(x) = k \frac{(x - y)}{|x - y|^3}.
\]

Consider for \( y \in \mathbb{R}^3 \) the delta distribution \( \delta_y \in D'(\mathbb{R}^3) \) defined by

\[
\langle \delta_y, \phi \rangle = \phi(y), \quad \phi \in C_c^\infty(\mathbb{R}^3).
\]

**(a)** Derive in the distributional sense \( \nabla \cdot D_y = k \delta_y, \) and find \( k. \)

**(b)** If \( D_N = \frac{1}{N} \sum_{i=1}^N D_{(i/N, 0, 0)} \) and \( \rho_N = k \frac{1}{N} \sum_{i=1}^N \delta_{(i/N, 0, 0)} \), derive distributions \( D \) and \( \rho \) such that \( D_N \to D \) and \( \rho_N \to \rho. \) Prove \( \nabla \cdot D = \rho \) in the distributional sense.

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**Category II: Fourier Analysis**

4. Consider the heat equation \( \frac{\partial}{\partial t} u(x, t) = \triangle u(x, t), \) and suppose \( u(x, 0) = u_0(x) \) where \( u_0 \in S(\mathbb{R}^d). \) Here \( u(x, t) \) is a scalar field.

**(a)** **(10 points)** Find the classical solution \( u(x, t) \) and verify that it is a classical solution, i.e. that \( u \) is twice continuously differentiable in \( x \) and continuously differentiable in \( t, \) and satisfies the PDE and initial data.

**(b)** **(10 points)** Now suppose \( u_0 \in L^2(\mathbb{R}^d). \) Use the same formula for your solution as in part (a) and show that \( u(x, t) \) satisfies classically the PDE \( \frac{\partial}{\partial t} u(x, t) = \triangle u(x, t) \) for \( t > 0. \)

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5. Consider \( \sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) smooth with sufficiently tempered growth. Suppose we define an operator \( Q \) acting on \( S(\mathbb{R}^d) \) by

\[
Q \psi(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma(x, \xi) \psi(y) e^{2\pi i (x-y) \cdot \xi} dy \, d\xi. \tag{3}
\]

**(a)** **(10 points)** Show that if \( \sigma(x, \xi) = q(x) + \xi^T A \xi \) for \( A \) a \( d \times d \) positive definite matrix, then \( Q \) can be written as a linear differential operator. Find that linear differential operator.
b. (10 points) Suppose \( Q\psi(x) = g(x) \cdot \nabla \psi(x) \) for some vector-valued continuous and bounded function \( g(x) \). Find \( \sigma(x, \xi) \) that satisfies Equation (3).

6. Let \( f \) be smooth and periodic, and consider the ODE
   \[
   -\frac{d^2}{dx^2}u(x) + 2\frac{d}{i\, dx}u(x) + u(x) = f(x).
   \]
   Find the solution \( u(x) \) in terms of the Fourier modes \( \hat{f}(n) = \int_{0}^{1} e^{2\pi i nx} f(x) dx \), and prove \( u \) is smooth and periodic.

Category III: Weak-form PDEs & Distribution Theory

7. Derive the distributional derivative of \( \ln |x| \) in \( \mathcal{D}'(\mathbb{R}) \).

8. Let \( \Omega = \{ x \in \mathbb{R}^3 : |x| \in (1, 2) \} \) and let \( H^1_0(\Omega) \) be defined as the completion of \( C_c^\infty(\Omega) \) with respect to the norm \( ||\phi||^2_1 = \int |\nabla \phi|^2 dx \). Consider the following PDE for \( f \in L^2(\Omega) \):
   \[
   -\Delta u(x) + |x|^2 u(x) = f(x), \quad x \in \Omega,
   \]
   \[
   u(x) = 0, \quad x \in \partial \Omega.
   \]
   Formulate the PDE in the weak sense, and prove the existence of a solution \( u \in H^1_0(\Omega) \).

9. a. (10 points) Prove the identity for \( \Phi, \Psi \in C^1(\mathbb{R}^3; \mathbb{R}^3) \):
   \[
   \nabla \cdot (\Psi \times \Phi) = \Phi \cdot (\nabla \times \Psi) - (\nabla \times \Phi) \cdot \Psi.
   \]

   b. (10 points) Consider simply connected domain \( \Omega = \Omega_1 \cup \Omega_2 \), where \( \Omega_1 \) and \( \Omega_2 \) are disjoint simply connected open sets. Let \( \Gamma \) be the boundary between \( \Omega_1 \) and \( \Omega_2 \). Consider the curl operator in the sense of distributions denoted \( \nabla \times \) on \( C_c^\infty(\Omega; \mathbb{R}^3) \). Let \( F \) be defined such that \( F(x) = F_j(x) \) for \( x \in \Omega_j \) where \( F_j \in C^2(\Omega_2; \mathbb{R}^3) \). Let \( [F](x) := F_2(x) - F_1(x) \) for \( x \in \Gamma \). Let \( N(x) \) be the normal vector for \( x \in \Gamma \) aiming into \( \Omega_2 \). Prove
   \[
   \nabla \times F = \nabla \times F_1 \bigg|_{\Omega_1} + \nabla \times F_2 \bigg|_{\Omega_2} + N \times [F] \delta_\Gamma.
   \]
   For a function \( G(x) \), the distribution \( G\delta_\Gamma \) is defined such that \( \langle G\delta_\Gamma, \Phi \rangle = \int_\Gamma G \cdot \Phi \).