Instructions: [*You must work two problems from each section for a total of four problems. Only your first four solutions will be graded.*]. *Be sure to write the number for each problem with your work and write your name clearly at the top of each page you turn in for grading. You have three hours.*

1 Point Set Topology

Problem 1 *Let $f : X \to \mathbb{R}$ be a continuous function.*
(i) If X is compact, show that f has a maximum value which it attains.
(ii) If X is connected and if $f(x) < c < f(z)$, show that there exists $y \in X$ such that $f(y) = c$.
(iii) If X is compact and connected, show that $f(X)$ is a point or a closed interval.

Problem 2 *Let C and each of C_i, $i \in I$ be connected subsets of X. If $C \cap C_i \neq \emptyset$ for each $i \in I$, prove or disprove that $C \cup \bigcup_i C_i$ is connected.*

Problem 3 *Let $f : X \to Y$ be a function between Hausdorff spaces. The* graph *of f is defined by*

$$Gr(f) = \{(x, y) \in X \times Y \,|\, y = f(x)\}.$$

(i) Show that if f is continuous and X is path-connected , then $Gr(f)$ is path-connected.
(ii) Show that if f is continuous, then $Gr(f)$ is closed.
(iii) Show that if $\overline{f(X)}$ is compact and $Gr(f)$ is closed, then f is continuous.

Problem 4 *The one-point compactification X^+ of a non-compact space X consists of the points of X together with a point $+$ not in X. The topology of X^+ consists of the open sets of X and all subsets U of X^+ such that $X^+ \setminus U$ is a closed compact subset of X.*
(i) Prove that X^+ is compact
(ii) Prove that X^+ is Hausdorff if and only if X is locally compact Hausdorff.
(iii) Sketch a picture representing the one-point compactification of each of the following spaces: (a) $X_1 = \mathbb{R}$; (b) $X_2 = (0, 1) \cup (1, 2) \subset \mathbb{R}$; (c) $[0, 1) \cup (1, 2) \cup (2, 3] \subset \mathbb{R}$; (d) $\mathbb{R}^2$.

2 Homotopy

Problem 5 *Let X be a locally path connected space with an open cover, $\{U, V\}$, consisting of two connected open sets such that U and V are both contractible and $U \cap V$ is connected.*
(i) Compute the fundamental group of X. State carefully any basic results you use.
(ii) Apply this result to compute $\pi_1(S^n)$ for $n > 1$.

Problem 6 *Let $\{f_i, g_i : i = 0, 1\}$ be four closed paths based at x_0. Define the concept of path-homotopy (denoted $\simeq$) and also the operation of path multiplication (denoted $f_0 \circ f_1$). Show that the following cancellation property holds for path-homotopy , if $f_0 \circ g_0 \simeq f_1 \circ g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.*

Problem 7 *Prove that the following three conditions (on a topological space X) are equivalent:*
(i) Every map $S^1 \to X$ is homotopic to a constant map.
(ii) Every map $S^1 \to X$ extends to a map $D^2 \to X$.
(iii) The fundamental group, $\pi_1(X, x_0)$, is trivial for all $x_0 \in X$.

Problem 8 *Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map, where E is path-connected. Let $H = p_*(\pi_1(E, e_0))$ be the image of the fundamental group of E in $\pi_1(B, b_0)$. Prove that there is a bijection from the set $\pi_1(B, b_0)/H$ of right cosets of H to the fiber $p^{-1}(b_0)$.*