Instructions:

1. *Use standard white 8.5 by 11 inch paper only. Put work on one side of each piece of paper and make sure that side is face up. (Work on the back of a page WILL NOT BE GRADED.) Write your name at the top of each page. Number each page and use a paper clip (not a staple) to hold your test together.*
2. *Write with a standard black pencil or black or dark blue ink only. Do not use a red or green pen.*
3. *Do any 6 of the 8 problems below. Indicate clearly which problems you have chosen.*
4. *You have three hours.*

Good luck!

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1. (a) Complete the following definition: Two topological spaces X and Y are homotopy equivalent if ......

   (b) Let X, Y, Z be topological spaces. Prove that if X is homotopy equivalent to Y, and Y is homotopy equivalent to Z, then X is homotopy equivalent to Z.

2. Let $f : X \to Y$ be a function from a topological space X to a space Y. We say that f is continuous if $f^{-1}(V)$ is open in X for every open subset V of Y. Show that f is continuous (in the preceding sense) if and only if for $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.

3. Give an example of a compact, connected space that is not path-connected, and prove that it has the stated properties.

4. Prove that the following subspace of $\mathbb{R}^3$,

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + x^2 = e^z\},$$

   is a smooth 2-dimensional manifold.

5. State precisely the Seifert-van Kampen theorem and use it to compute the fundamental group of the connected sum of two projective planes: $\mathbb{RP}^2 \# \mathbb{RP}^2$.

6. Give an example of a space X whose fundamental group is $\pi_1(X, x_0) = \{a, b \mid a^2ba^{-2}b^{-1} = 1\}$. Provide a proof that your space has the specified fundamental group.

7. Let $p : \tilde{X} \to X$ be a covering space with path-connected cover. Explicitly define the right action of $\pi_1(X, x)$ on the fiber $p^{-1}(x)$ for a given point $x \in X$. Show that this is a group action that satisfies the transitive property.

8. Suppose that X and Y are topological spaces, and let $x_0 \in X$ and $y_0 \in Y$. Prove that

$$\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$
