Instructions: *Work two problems from section A, two problems from section B, and one problem from section C, for a total of five problems.* Be sure to write the number for each problem with your work, and write your name clearly at the top of each page you turn in for grading. You have three hours.

A Point Set Topology (2 problems)

A1. Let X be a topological space with the property that for any two distinct points of X, there is an open set containing exactly one of them. Suppose also that for any $x \in X$ and closed subset A of X not containing x, there are disjoint open sets $U \ni x$ and $V \supseteq A$. Prove that X is Hausdorff.

A2. Let $p\colon X \to Y$ be a quotient map, and $f\colon X \to Z$ a continuous function such that $f(x_1) = f(x_2)$ whenever $p(x_1) = p(x_2)$. Show that there is a unique function $g\colon Y \to Z$ such that $g \circ p = f$, and that g is continuous.

A3. Let $p\colon X \to Y$ be a closed, continuous surjection.

(a) Prove that if $y \in Y$ and $U \subseteq X$ is an open set containing $p^{-1}(y)$, there is a neighborhood V of y such that $p^{-1}(V) \subseteq U$.

(b) Prove that if Y is compact and $p^{-1}(y)$ is compact for every $y \in Y$, then X is compact.

B Homotopy (2 problems)

B1. Let $p\colon E \to B$ be a covering map with connected base space B, and let x and y be two points in B. Show that the sets $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality.

B2. Let X be the subspace of $\mathbb{R}^2$ that is the union of two circles of radius 1 centered at $(-2, 0)$ and $(2, 0)$, and the line segment from $(-1, 0)$ to $(1, 0)$. State the Seifert-van Kampen Theorem, and use it to find the fundamental group of X.

B3. Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map, where E is path connected. Let $H = p_*(\pi_1(E, e_0))$ be the image of the fundamental group of E in $\pi_1(B, b_0)$. Prove that there is a bijection from the set $\pi_1(B, b_0)/H$ of right cosets of H to the fiber $p^{-1}(b_0)$.

C Mixed (1 problem)

C1. (a) Let X be a Hausdorff space and A a compact subset of X. Prove that A is closed.

(b) Let $f\colon X \to Y$ be a continuous map from a compact space X to a Hausdorff space Y. Prove that f is a closed map.

C2. A subset X of $\mathbb{R}^n$ is *convex* if the line segment between any two points of X is contained in X; it is *star-convex* if there is some point x_0 in X such that the line segment from x_0 to any other point of X is contained in X. Give an example of a star-convex set that is not convex, and prove that any star-convex set is contractible.