Instructions: *Work two problems from Section A, two problems from Section B, and one problem from Section C, for a total of five problems.* Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

A. **Point Set Topology** (2 problems)

A1. Let $f : S^1 \to \mathbb{R}$ be continuous, where S^1 is the unit circle in $\mathbb{R}^2$.

i. Show that there is a point $x \in S^1$ such that $f(x) = f(-x)$

ii. Show that f is not surjective.

A2. i. Let X be a Hausdorff space and A be a compact subset of X. Prove that A is closed.

ii. Let $f : X \longrightarrow Y$ be a continuous map from a compact space X to a Hausdorff space Y . Prove that f is a closed map.

A3. Let $f : X \to Y$ be a continuous function between topological spaces X and Y, and let A be a subset of X.

i. If A is compact, prove that $f(A)$ is compact.

ii. If A is connected, prove that $f(A)$ is connected.

B. **Homotopy** (2 problems)

B1. Let $p : E \to B$ be a covering map with connected base space B, and let x and y be two points in B. Show that the sets $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality.

B2. Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$.

B3. Let A be a path connected subspace of a space X, and $a_0 \in A$. Show that the inclusion of A in X induces a surjection from $\pi_1(A, a_0)$ to $\pi_1(X, a_0)$ if and only if every path in X with endpoints in A is path homotopic to a path in A.

C. **Mixed** (1 problem)

C1. Show that the following three conditions on a topological space X are equivalent:

i. Every continuous map $S^1 \longrightarrow X$ is null homotopic.

ii. Every continuous map $S^1 \longrightarrow X$ extends to a continuous map $D^2 \longrightarrow X$.

iii. The fundamental group $\pi_1(X, x_0)$ is trivial for all $x_0 \in X$.

C2. Let $X = S^1 \vee S^1$ be a wedge of two circles (a figure eight - two circles joined at a single point).

i. Determine the fundamental group of X using the fact that the fundamental group of S^1 is isomorphic to the integers, $\mathbb{Z}$.

ii. Give the definition of a covering space, and explicitly construct a 3-fold covering space of X.
