Instructions: *Work two problems from Section A, two problems from Section B, and one problem from Section C, for a total of five problems.* Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

A. **Point Set Topology** (2 problems)

A1. Let $f : S^1 \to \mathbb{R}$ be continuous, where S^1 is the unit circle in $\mathbb{R}^2$.

i. Show that there is a point $x \in S^1$ such that $f(x) = f(-x)$

ii. Show that f is not surjective.

A2. Recall that a separation of a topological space X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be connected if there does not exist a separation of X.

i. If the open sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.

ii. Show that the union of a collection of connected subspaces of X that have a point in common is connected.

A3. i. Let X be a Hausdorff space and A be a compact subset of X. Prove that A is closed.

ii. Let $f : X \longrightarrow Y$ be a continuous map from a compact space X to a Hausdorff space Y . Prove that f is a closed map.

B. **Homotopy** (2 problems)

B1. Let $p : E \to B$ be a covering map with connected base space B, and let x and y be two points in B. Show that the sets $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality.

B2. Prove that $\mathbb{R}^2$ is not homeomorphic to the torus, and to $\mathbb{R}^n$ for $n > 2$.

B3. Let A be a path connected subspace of a space X, and $a_0 \in A$. Show that the inclusion of A in X induces a surjection from $\pi_1(A, a_0)$ to $\pi_1(X, a_0)$ if and only if every path in X with endpoints in A is path homotopic to a path in A.

C. **Mixed** (1 problem)

C1. Show that the following three conditions on a topological space X are equivalent:

i. Every continuous map $S^1 \longrightarrow X$ is null homotopic.

ii. Every continuous map $S^1 \longrightarrow X$ extends to a continuous map $D^2 \longrightarrow X$.

iii. The fundamental group $\pi_1(X, x_0)$ is trivial for all $x_0 \in X$.

C2. Prove the Brouwer fixed point theorem in dimensions one and two: Every continuous map $f : D^n \longrightarrow D^n$, $n = 1, 2$, has a fixed point.
