Instructions: *Work two problems from Section A, two problems from Section B, and one problem from Section C, for a total of five problems.* Start each problem on a new page, and write the problem number on the **top right** corner of the page. Make sure you order the pages correctly before submitting the exam. You have three hours. Good luck!

A. **Point Set Topology** (2 problems)

A1. i. Show that if $f : X \to Y$ is a continuous bijection from a compact space X to a Hausdorff space Y, then f is a homeomorphism.

ii. Give an example (with justification) of topological spaces X and Y, and a continuous bijection $f : X \to Y$ that is not a homeomorphism. Which hypotheses in (i) do your examples fail?

A2. i. Define what it means for a topological space to be *connected*.

ii. Using only the definition of connectedness, prove the following:
Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of connected subsets of a topological space X. Suppose that $Y_n \cap Y_{n+1}$ is nonempty for all n. Show that $\cup_n Y_n$ is connected.

A3. Let X and Y be topological spaces, and let $p : X \to Y$ be a closed, continuous surjection.

i. Prove: Given any $y \in Y$ and any open set U in X containing $p^{-1}(y)$, there is a neighborhood V of y in Y such that $p^{-1}(V) \subseteq U$.

ii. Prove that if Y is compact and $p^{-1}(y)$ is compact for every $y \in Y$, then X is compact.

B. **Algebraic Topology** (2 problems)

B1. Let $p : \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ path-connected. Explicitly define the action of $\pi_1(X, x)$ on the fiber $p^{-1}(x)$ for a given point $x \in X$. Show that this is a group action which is transitive (i.e., given any pair of elements in $p^{-1}(x)$, there exists a group element taking one to the other).

B2. Let X be the space obtained by gluing a disk D^2 to a torus $T^2 = S^1 \times S^1$ by identifying the boundary of D^2 with the loop $\alpha : [0, 1] \to T^2$ given by $\alpha(t) = (4\pi t, 0)$ (i.e. α is a curve that wraps around the first S^1 factor twice). More precisely, if $\beta : [0, 1] \to \partial D^2 = S^1$ is the boundary loop of D^2, then X is the quotient space

$$X = \left(T^2 \sqcup D^2\right) \Big/ \alpha(t) \sim \beta(t) \; \forall t \in [0, 1].$$

State the Seifert–van Kampen Theorem and use it to find the fundamental group of X. Make sure to verify all the hypotheses of the theorem. Write your final answer as a presentation.

B3. For each pair of spaces (X, A) in the following list, determine whether or not a retraction $r : X \to A$ exists.

- If a retraction exists, construct it as explicitly as possible and demonstrate that it is a retraction.
  Is your construction a deformation retraction (no justification needed for this part)?
- If a retraction does not exist, prove it.

(a) $X = \mathbb{R}^2 \setminus (0, 0)$, the punctured plane and $A = (0, 1)$, a single point.

(b) $X = D^2 = \{(x, y) \in R^2 \mid x^2 + y^2 \leq 1\}, A = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

(c) X is $\mathbb{R}^3$ with the three coordinate axes removed, and A is the unit sphere in $\mathbb{R}^3$ with the 6 points $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$ removed.

C. **Mixed** (1 problem)

C1. The suspension of a topological space X is the quotient space $\Sigma X = X \times [0, 1]/\sim$ where $(x, t) \sim (y, s)$ if and only if either $(x, t) = (y, s)$ or $s = t = 1$ or $s = t = 0$.

i. Prove that the suspension ΣX need not be simply connected.

ii. Prove that the suspension ΣX is simply connected if X is path connected. (Hint: Use the Seifert–van Kampen Theorem. Make sure to verify all the hypotheses of the theorem.)

C2. Let $\mathbb{R}P^2$ denote the real projective plane.

i. Compute the fundamental group of $\mathbb{R}P^2$. (You may assume a result about fundamental groups of CW complexes).

ii. Show that every continuous map $f : \mathbb{R}P^2 \to S^1$ is null-homotopic. (Hint: use covering space theory.)