

Instructions: Work two problems from Section A, two problems from Section B, and one problem from Section C, for a total of five problems. Start each problem on a new page, and write the problem number in the **upper-right** corner of the page. Place your completed pages in numerical order before submitting the exam. You have three hours. Good luck!

A. Point Set Topology (2 problems)

A1. Let A be a subset of a topological space X . Prove or give a counterexample:

- i. If A is connected, then its closure $\overline{A}$ is also connected.
- ii. If A is connected, then its interior A° is also connected.

A2. Let $f : X \rightarrow Y$ be a continuous surjection between topological spaces.

- i. If X is connected, prove that Y is also connected.
- ii. If X is compact, prove that Y is also compact.

A3. Show that a topological space X is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) \mid x \in X\} \subset X \times X$$

is a closed subset of $X \times X$.

B. Algebraic Topology (2 problems)

B1. Compute the fundamental group of $\mathbb{R}^3 \setminus A$, where

$$A = \{(x, 0, 0) \mid x \geq 0\} \cup \{(0, y, 0) \mid y \geq 0\} \cup \{(0, 0, z) \mid z \geq 0\}.$$

Justify your computation and recall any results you are using.

B2. Let $p : E \rightarrow B$ be a covering map, where B is path-connected. Prove that for any $x, y \in B$, there is a bijection

$$p^{-1}(x) \rightarrow p^{-1}(y).$$

Conclude that $|p^{-1}(x)|$ is independent of $x \in B$.

- B3. i. Use the Seifert–van Kampen theorem to compute the fundamental group of S^n for $n \geq 2$.
- ii. Prove that S^1 is not homotopy equivalent to S^n for $n \geq 2$.
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C. Mixed (1 problem)

- C1. i. Suppose that $j : A \hookrightarrow X$ is the inclusion map of the subspace $A \subseteq X$ and that A is a retract of X . Prove that, for $x_0 \in A$, the induced homomorphism $j_* : \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$ is injective.
- ii. Prove that there is no retraction from the solid torus $D^2 \times S^1$ to its boundary $S^1 \times S^1$. Here S^1 is the unit circle, and D^2 is the unit disk in the plane $\mathbb{R}^2$.
- C2. Assume in this problem that X and Y are path-connected. Which of the following are possible? Justify your answer in each case. When a situation is possible, construct an example.
- i. The covering transformation group of a 6-sheeted covering map $p : X \rightarrow Y$ can have 3 elements.
- ii. The covering transformation group of a 10-sheeted covering map $p : X \rightarrow Y$ can have 4 elements.
- iii. The covering transformation group of an 8-sheeted regular covering map $p : X \rightarrow Y$ can have 4 elements.
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