Instructions: *Work two problems from Section A, two problems from Section B, and one problem from Section C, for a total of five problems.* Start each problem on a new page, and write the problem number on the **top right** corner of the page. Make sure you order the pages correctly before submitting the exam. You have three hours. Good luck!

A. **Point Set Topology** (2 problems)

A1. Let X and Y be topological spaces. Recall that a function $f : X \longrightarrow Y$ is *open* if for any open subset U of X, the image $f(U)$ is open in Y.

i. If f is continuous, does it follow that f is open? (Prove or give a counterexample.)

ii. If f is open, does it follow that f is continuous? (Prove or give a counterexample.)

iii. Suppose that $X = Y \times Z$ is a product of topological spaces equipped with the product topology, and that f is the first projection. Show that f is open.

iv. Suppose that $X = Y \times Z$ is a product of topological spaces equipped with the product topology, and that f is the first projection. If A is a closed subset of X, does it follow that $f(A)$ is closed in Y? (Prove or give a counterexample.)

A2. Let X be a topological space and let $\{A_\alpha\}$ be a nonempty collection of connected subsets of X such that

$$\bigcap_\alpha A_\alpha \neq \varnothing.$$

Show that the union $\bigcup_\alpha A_\alpha$ is connected.

A3. i. Let X and Y be topological spaces and suppose that X is compact and Y is Hausdorff. Show that if $f : X \longrightarrow Y$ is a continuous bijection, then f is a homeomorphism.

ii. Give an example of two topological spaces X and Y and a continuous bijection $f : X \longrightarrow Y$ that is **not** a homeomorphism.

B. **Algebraic Topology** (2 problems)

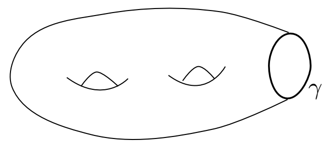
B1. Calculate the fundamental group of $\mathbb{R}^3 \setminus X$, where X is the space

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0; y = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1; z = 0\}.$$

Justify your computation and recall any results you are using.

B2. Let X, Y, and Z be path connected, locally path connected spaces, and let $p : X \to Z$, $q : X \to Y$, and $r : Y \to Z$ be continuous maps with $p = r \circ q$. If $p : X \to Z$ and $r : Y \to Z$ are covering maps, prove that $q : X \to Y$ is also a covering map.

B3. Let M be the surface of genus 2 with one boundary component, pictured below:



i. Find a CW complex structure for M and use it to compute $\pi_1(M)$. Write your answer as a presentation.

ii. Prove that there is no retraction from M to γ, where γ is the boundary curve. (Hint: Use $\pi_1(M)$.)

C. **Mixed** (1 problem)

C1. Prove that no two of the following three spaces are homeomorphic:

i. $\mathbb{R}^2$

ii. $\mathbb{R}^3$

iii. The unit sphere S^2

C2. Let $f : \mathbb{RP}^2 \to S^1 \times S^1$ be continuous.

i. Determine f_*, the induced map on fundamental groups.

ii. Prove that f is null homotopic.