Topology I. Comprehensive Exam, August, 2002.

Solve the starred problems (i.e., number 1 and 2.) Then choose two problems from the remaining four, and solve them. Total required: 4 problems. Partial credit will be given for relevant work, but do not return more than 4 problems (as indicated.) Please start each problem on a new page, and be as clear as possible. You have 2 and 1/2 hours for this test. Thanks, and good luck!

(*) 1. (a) Show that any open interval (a, b) in the real line $\mathbf{R}$ (where $a < b$ are real numbers) is homeomorphic to the real line $\mathbf{R}$.

(b) Show that $\mathbf{R}$ is not homeomorphic to $\mathbf{R}^2$. (*All the spaces involved have the usual topologies, quote without proof the appropriate basic results you use.)*

(*) 2. Let X and Y be topological spaces, $f : X \to Y$ a function. Recall that "f is open" means: if U is an open set in X then $f(U)$ is open in Y.

(a) If f is continuous, does it follow that f is open? (Proof or counterexample.)

(b) If f is open, does it follow that f is continuous? (Proof or counterexample.)

(c) Show that if $X = Y \times Z$ (a product of topological spaces, with the product topology) and f is the first projection, then f is open.

(d) Under the conditions and notation of (c), if F is closed in $X = Y \times Z$, does it follow that $f(F)$ is closed in Y? (Proof or counterexample.)

3. Let $I = [0, 1]$ be the closed unit interval in the real line $\mathbf{R}$, with the usual topology, J any set, $X = I^J$ with the product topology, $f : X \to \mathbf{R}$ a continuous function. Show that for each point $c \in \mathbf{R}$ the fiber $f^{-1}(c)$, regarded as a subspace of X, is compact. *[Quote without proof the appropriate standard topological results.]*

4. Let $f : X \to Y$ be a quotient map of topological spaces, such that Y is connected and each set $f^{-1}(y)$, $y \in Y$, is a connected subspace of X. Show that X is connected.
[Recall that a mapping $f : X \to Y$ of topological spaces is a *quotient map* if f is onto (or surjective) and a subset U of Y is open if and only if $f^{-1}(U)$ is open.]

5. Let $f : S^1 \to \mathbf{R}$ be a continuous function (where S^1, the "unit circle", is the set of all complex numbers of modulus one.).

(a) Show there is a point z of S^1 such that $f(z) = f(-z)$.

(b) Show that this function f cannot be onto (or surjective). *[Quote without proof the appropriate standard topological results.]*

6. Recall that a metric d on a set X is called *bounded* if there is a positive real constant M such that $d(x, y) \leq M$ for any pair of points x, y in X. Show that given any metric δ on a set X, there is a bounded metric d on X that induces the same topology as δ.