
Solve the starred problems (i.e., number 1 and 2.) Then choose two problems from the remaining four, and solve them. Total required: 4 problems. Partial credit will be given for relevant work, but do not return more than 4 problems (as indicated.) Please start each problem on a new page, and be as clear as possible. You have 2 and 1/2 hours for this test. Thanks, and good luck!

(*) 1. (a) Show that any open interval \((a, b)\) in the real line \(\mathbb{R}\) (where \(a < b\) are real numbers) is homeomorphic to the real line \(\mathbb{R}\).
(b) Show that \(\mathbb{R}\) is not homeomorphic to \(\mathbb{R}^2\). (All the spaces involved have the usual topologies, quote without proof the appropriate basic results you use.)

(*) 2. Let \(X\) and \(Y\) be topological spaces, \(f : X \to Y\) a function. Recall that “\(f\) is open” means: if \(U\) is an open set in \(X\) then \(f(U)\) is open in \(Y\).
(a) If \(f\) is continuous, does it follow that \(f\) is open? (Proof or counterexample.)
(b) If \(f\) is open, does it follow that \(f\) is continuous? (Proof or counterexample.)
(c) Show that if \(X = Y \times Z\) (a product of topological spaces, with the product topology) and \(f\) is the first projection, then \(f\) is open.
(d) Under the conditions and notation of (c), if \(F\) is closed in \(X = Y \times Z\), does it follow that \(f(F)\) is closed in \(Y\)? (Proof or counterexample.)

3. Let \(I = [0, 1]\) be the closed unit interval in the real line \(\mathbb{R}\), with the usual topology, \(J\) any set, \(X = I^J\) with the product topology, \(f : X \to \mathbb{R}\) a continuous function. Show that for each point \(c \in \mathbb{R}\) the fiber \(f^{-1}(c)\), regarded as a subspace of \(X\), is compact. [Quote without proof the appropriate standard topological results.]

4. Let \(f : X \to Y\) be a quotient map of topological spaces, such that \(Y\) is connected and each set \(f^{-1}(y), y \in Y\), is a connected subspace of \(X\). Show that \(X\) is connected. [Recall that a mapping \(f : X \to Y\) of topological spaces is a quotient map if \(f\) is onto (or surjective) and a subset \(U\) of \(Y\) is open if and only if \(f^{-1}(U)\) is open.]

5. Let \(f : S^1 \to \mathbb{R}\) be a continuous function (where \(S^1\), the “unit circle”, is the set of all complex numbers of modulus one.).
(a) Show there is a point \(z\) of \(S^1\) such that \(f(z) = f(-z)\).
(b) Show that this function \(f\) cannot be onto (or surjective). [Quote without proof the appropriate standard topological results.]

6. Recall that a metric \(d\) on a set \(X\) is called bounded if there is a positive real constant \(M\) such that \(d(x, y) \leq M\) for any pair of points \(x, y\) in \(X\). Show that given any metric \(\delta\) on a set \(X\), there is a bounded metric \(d\) on \(X\) that induces the same topology as \(\delta\).
