Core I Topology Exam

August, 2003

Instructions: Do problems 1* and 2* and any two others for a total of four problems. Put your name on the top right corner of each sheet, and write on only one side of each sheet. Copy each problem before solving it. Turn in ONLY the four problems you want graded. *Think carefully about your write-up; clarity, logic, and completeness of your solutions will be considered when the problems are graded.* You have 2 and 1/2 hours. GOOD LUCK!

1*. Let $F : X \longrightarrow Y$ be a continuous function between topological spaces X, Y, and let A be a subset of X.

a. If A is compact, prove that $f(A)$ is compact.

b. If A is connected, prove that $f(A)$ is connected.

2*. Let X, Y be topological spaces, $X = A \cup B$, for closed sets A and B, and $f : X \longrightarrow Y$ be a function such that the restrictions $f_{\big|A}$ and $f_{\big|B}$ are continuous with respect to the relative topologies on A, B, respectively. Prove that f is continuous.

3. Let $Y = \{(0, y) \in R^2 : -1 \leq y \leq 1\}$ and let Z be the graph of the function $y = \sin(\pi/x)$ for $0 < x \leq 1$. Is the union $Y \cup Z$ connected or disconnected in the standard topology on R^2? Prove your answer.

4.

a. Let $f : X \longrightarrow Y$ be a continuous map from a compact space S to a Hausdorff space Y. Prove that f is a closed map.

b. Let X be a Hausdorff space and A be a compact subset of S. Prove that A is closed.

5. Prove that every finite Hausdorff space is discrete.

6. Let (X, d) be a complete metric space (a metric space in which Cauchy sequences converge), and let $f : X \longrightarrow X$ be a function for which $d(f(x), f(y)) \leq \frac{1}{2} d(x, y)$ for all $x, y \in X$. Prove that f has exactly one fixed point.