Comprehensive Exam **Core I Topology** **August 2005**

Instructions: *Do* **any four** *of the following problems. Indicate clearly which problems you have chosen. You have* $2\frac{1}{2}$ *hours to finish the test.* Good luck!

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1. Show that a metrizable space is normal.

2. Let $f: X \to Y$ be a function between topological spaces. The *graph* of f is defined by

$$G_f = \{(x, y) \in X \times Y \mid y = f(x)\}.$$

(i) Show that if Y is Hausdorff and f is continuous, then G_f is closed.
(ii) Show that if Y is compact and G_f is closed, then f is continuous. (You may use the fact that $\pi_1: X \times Y \to X$ is a closed map when Y is compact.)

3. Let $p: X \to Y$ be a quotient map. Show that if Y is connected and $p^{-1}(y)$ is connected for each $y \in Y$, then X is connected.

4. Let $f: X \to Y$ be a function from a topological space X to a space Y. We say that f is continuous if $f^{-1}(V)$ is open in X for every open subset V of Y. Show that f is continuous (in the preceding sense) if and only if for all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

5. (i) Show that if $f: X \to Y$ is a continuous bijection from a compact space X to a Hausdorff space Y, then f is a homeomorphism.
(ii) Give an example of topological spaces X and Y and a continuous bijection $f: X \to Y$ that is **not** a homeomorphism.

6. Let $Y = \{(0, y) \in \mathbb{R}^2 \mid -1 \le y \le 1\}$ and let Z be the graph of the function $y = \sin(\pi/x)$ for $0 < x \le 1$. Is the set $X = Y \cup Z$ connected or disconnected in the standard topology on $\mathbb{R}^2$? Prove your answer.
