Instructions: Do any four of the following problems. Indicate clearly which problems you have chosen. You have $2\frac{1}{2}$ hours to finish the test. Good luck!

1. Show that a metrizable space is normal.

2. Let $f: X \to Y$ be a function between topological spaces. The graph of $f$ is defined by
   
   $$G_f = \{(x, y) \in X \times Y \mid y = f(x)\}.$$
   
   (i) Show that if $Y$ is Hausdorff and $f$ is continuous, then $G_f$ is closed.
   
   (ii) Show that if $Y$ is compact and $G_f$ is closed, then $f$ is continuous. (You may use the fact that $\pi_1: X \times Y \to X$ is a closed map when $Y$ is compact.)

3. Let $p: X \to Y$ be a quotient map. Show that if $Y$ is connected and $p^{-1}(y)$ is connected for each $y \in Y$, then $X$ is connected.

4. Let $f: X \to Y$ be a function from a topological space $X$ to a space $Y$. We say that $f$ is continuous if $f^{-1}(V)$ is open in $X$ for every open subset $V$ of $Y$. Show that $f$ is continuous (in the preceding sense) if and only if for all $A \subseteq X$, $f(A) \subseteq f(A)$.

5. (i) Show that if $f: X \to Y$ is a continuous bijection from a compact space $X$ to a Hausdorff space $Y$, then $f$ is a homeomorphism.

   (ii) Give an example of topological spaces $X$ and $Y$ and a continuous bijection $f: X \to Y$ that is not a homeomorphism.

6. Let $Y = \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$ and let $Z$ be the graph of the function $y = \sin(\pi/x)$ for $0 < x \leq 1$. Is the set $X = Y \cup Z$ connected or disconnected in the standard topology on $\mathbb{R}^2$? Prove your answer.