Core I Topology Exam

August, 2004

Instructions: Do problems 1* and 2* and any two others for a total of four problems. Put your name on the top right corner of each sheet, and write on only one side of each sheet. Copy each problem before solving it. Turn in ONLY the four problems you want graded. Think carefully about your write-up; clarity, logic, and completeness of your solutions will be considered when the problems are graded. GOOD LUCK!

1*. Let X be a topological space, let A be a closed subset of X and let B be a subset of A that is closed in the relative topology on A inherited from X. Prove that B is closed as a subset of X.

2* Let $f : A \longrightarrow B$ be a continuous function between topological spaces.
a. If A is compact, prove that $f(A)$ is compact.
b. If A is connected, prove that $f(A)$ is connected.

3. Let X, Y be topological spaces, $X = A \cup B$ where A, B are closed subsets of X, and $f : X \longrightarrow Y$ be a function such that the restrictions $f\big|_A$ and $f\big|_B$ are continuous with respect to the relative topologies on A, B, respectively. Prove that f is continuous.

4. Let X be a topological space, let E be a connected subspace, and let A be a subset of X sandwiched between E and its closure $\overline{E}$:

$$E \subset A \subset \overline{E}.$$

Prove that A is a connected subset of X. Show every step; do not just say "This is a theorem".

5. a. Let X be a Hausdorff space and A be a compact subset of X. Prove that A is closed.
b. Let $f : X \longrightarrow Y$ be a continuous map from a compact space X to a Hausdorff space Y. Prove that f is a closed map.

6. Let (X, d) be a complete metric space (a metric space in which Cauchy sequences converge), and let $f : X \longrightarrow X$ be a function for which $d(f(x), f(y)) \leq \frac{1}{2} d(x, y)$ for all $x, y \in X$. Prove that f has exactly one fixed point.