Instructions: [*You must work two problems from each section for a total of four problems. Only your first four solutions will be graded.*]. *Be sure to write the number for each problem with your work and write your name clearly at the top of each page you turn in for grading. You have three hours.* Good luck!

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1 Point Set Topology

1. Prove that a path-connected space is connected. Then, prove that a connected, locally path-connected space is path-connected.

2. Let $f : S^1 \to \mathbb{R}$ be continuous, where S^1 is the unit circle in $\mathbb{R}^2$.

   (a) Show that there is a point $z \in S^1$ such that $f(z) = f(-z)$.

   (b) Show that f is not surjective.

3. Prove that a function $f : X \to Y$ between metric spaces is continuous if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$ is a convergent sequence in X.

4. Let $p : X \to Y$ be a quotient map, and $f : X \to Z$ a continuous function such that $f(x_1) = f(x_2)$ whenever $p(x_1) = p(x_2)$. Prove that there is a unique function $g : Y \to Z$ such that $g \circ p = f$, and that g is continuous.

2 Homotopy

5. Let $\{f_i, g_i : i = 0, 1\}$ be four closed paths based at x_0. Define the concept of path-homotopy (denoted $\simeq$) and also the operation of path multiplication (denoted $f_0 \circ f_1$). Show that the following cancellation property holds for path-homotopy , if $f_0 \circ g_0 \simeq f_1 \circ g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.

6. Prove the Brouwer fixed point theorem in dimension two: Every continuous map $f : D^2 \to D^2$ has a fixed point.

7. Let X be a locally path connected space with an open cover, $\{U, V\}$, consisting of two connected open sets such that U and V are both contractible and $U \cap V$ is connected. Compute its fundamental group. State carefully any basic results you use.

8. Let $X = S^1 \vee S^1$, the join of two circles joined at a single point. Compute the fundamental group of X from the fact that the fundamental group of S^1 is isomorphic to the integers, $\mathbb{Z}$. Define *covering space* and explicitly construct a 3-fold covering space of X.