Instructions:

1. *Use standard white 8.5 by 11 inch paper only. Put work on one side of each piece of paper and make sure that side is face up. (Work on the back of a page WILL NOT BE GRADED.) Write your name at the top of each page. Number each page and use a paper clip (not a staple) to hold your test together.*

2. *Write with a standard black pencil or black or dark blue ink only. Do not use a red or green pen.*

3. *Do any 7 of the 9 problems below. Indicate clearly which problems you have chosen.*

4. *You have three hours.*

Good luck!

1. Let $\{f_i, g_i : i = 0, 1\}$ be four closed paths in a space X based at a point x_0 in X.
   (a) Define the concept of path-homotopy (denoted $\simeq$).
   (b) Show that the following cancellation property for path multiplication holds for path-homotopy: if $f_0 \star g_0 \simeq f_1 \star g_1$ and $g_0 \simeq g_1$, then $f_0 \simeq f_1$.

2. Let S denote the following union of three sets in $\mathbb{R}^2$:
$$S = \{(t, 0) \,|\, 0 < t \leq 1\} \cup \{(\frac{1}{n}, s) \,|\, n = 1, 2, 3, \ldots \text{ and } 0 \leq s \leq 1\} \cup \{(0, \frac{1}{2})\}.$$
   State whether S is connected or not, and prove your assertion.

3. Let $p : X \to Y$ be a closed, continuous surjection. Prove that if Y is compact and $p^{-1}(y)$ is compact for every $y \in Y$, then X is compact.

4. (a) Prove that a path-connected space is connected.
   (b) Prove that a connected, locally path-connected space is path-connected.

5. Prove that the following subspace of $\mathbb{R}^4$,
$$C = \{(w, x, y, z) \,|\, w^2 + x^2 = 1,\ y^2 + z^2 = 1\} \subset \mathbb{R}^4,$$
   is a smooth 2-dimensional manifold. *(Hint: You can either use the definition of a smooth manifold directly or use a theorem that is an application of Sard's theorem.)*

6. For the map $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = y^2 + z^2 - (1 + x^2)^2$,
   (a) Prove that zero is a regular value of f.
   (b) Draw a picture of $f^{-1}(0)$ in $\mathbb{R}^3$.
   (c) Let $M = f^{-1}(0)$ and let $N = \{(x, y, z) \in \mathbb{R}^3 \,|\, x = 0\}$. Prove that for all $p \in M \cap N$,
$$T_pM + T_pN = T_p\mathbb{R}^3$$
   using paths through p in both M and N.

7. Let $X = S^1 \vee S^1$, the join of two circles joined at a single point x_0.
   (a) Compute the fundamental group $\pi_1(X, x_0)$ of X using Seifert-Van Kampen's theorem.
   (b) Construct (illustrate) a regular 3-fold covering space $\tilde{X}$ of X.
   (c) Exhibit a presentation for $\pi_1(\tilde{X}, e_0)$ and describe how the map $p : \tilde{X} \to X$ maps each of the generators of $\pi_1(\tilde{X}, e_0)$ to $\pi_1(X, x_0)$ via $p_\# : \pi_1(\tilde{X}, e_0) \to \pi_1(X, x_0)$.

8. Let X be a closed surface of genus two.
   (a) Using Seifert-Van Kampen's theorem, compute the fundamental group of X.
   (b) Assuming the classification of finitely generated free abelian groups, prove that X is not homeomorphic a torus.

9. Let $p : \tilde{X} \to X$ be a covering space (both $\tilde{X}$ and X are path connected, locally path-connected). Explicitly define the right action of $\pi_1(X, x)$ on the fiber $p^{-1}(x)$ for a given point $x \in X$. Show that this is a group action that satisfies the transitive property.