Instructions: *Work two problems from each section for a total of four problems.* Be sure to write the number for each problem with your work, and write your name clearly at the top of each page you turn in for grading. You have three hours.

A Point Set Topology

A1. Let $f\colon X \to Y$ be a function between topological spaces. The *graph* of f is the subset

$$G_f = \{x \times f(x) \mid x \in X\}$$

of $X \times Y$.

(a) Show that if Y is Hausdorff and f is continuous, then G_f is closed.

(b) Show that if Y is compact and G_f is closed, then f is continuous. (You may use the fact that the projection $\pi_1\colon X \times Y \to X$ is a closed map when Y is compact.)

A2. Prove that if A is a connected subspace of a topological space X then the closure of A is connected. Give an example where A is connected, but the interior of A is not connected.

A3. Prove that a path-connected space is connected, and that a space that is connected and locally path-connected is path-connected.

A4. Let X be a space with a countable basis. Prove that every open cover of X has countable subcover.

B Homotopy

B1. Let X be a path-connected space, and let $f\colon X \to Y$ be a continuous map with $f(x_0) = y_0$ and $f(x_1) = y_1$. Let $f_{*0}\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $f_{*1}\colon \pi_1(X, x_1) \to \pi_1(Y, y_1)$ be the induced homomorphisms on fundamental groups. Show that there are isomorphisms $\phi\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ and $\psi\colon \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ such that the following diagram commutes.

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_{*0}"] \arrow[d, "\phi"] & \pi_1(Y, y_0) \arrow[d, "\psi"] \\
\pi_1(X, x_1) \arrow[r, "f_{*1}"] & \pi_1(Y, y_1)
\end{tikzcd}$$

B2. Prove that a retract of a contractible space is contractible.

B3. Prove that for any topological spaces X and Y and points $x_0 \in X$ and $y_0 \in Y$,

$$\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

B4. Let X be a topological space and x_0 a point of X such that $\{x_0\}$ is a deformation retract of X. Show that for any neighborhood U of x_0, the path-component of U containing x_0 contains a neighborhood of x_0. (Hint: use the tube lemma.)