Instructions: *Work 2 problems from section A, 2 problems from section B, and 1 problem from section C.* Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours.

A Point Set Topology (2 problems)

A1. Let X and Y be topological spaces, and $f\colon X \to Y$ a function. If $X = A \cup B$, where A and B are closed subsets of X, and the restrictions $f|_A$ and $f|_B$ are continuous with respect to the subspace topologies on A and B, prove that f is continuous.

A2. (a) Prove that a path connected space is connected.

(b) Prove that a connected, locally path connected space is path connected.

A3. A topological space X is normal if for each pair A, B of disjoint closed sets in X, there exist disjoint open sets containing A and B. Show that a metric space is normal.

B Homotopy (2 problems)

B1. Let A be a path connected subspace of a space X, and $a_0 \in A$. Show that the inclusion of A in X induces a surjection from $\pi_1(A, a_0)$ to $\pi_1(X, a_0)$ if and only if every path in X with endpoints in A is path homotopic to a path in A.

B2. Show that if $n > 1$, every continuous map $f\colon S^n \to S^1$ is nullhomotopic.
[*Hint:* Use the general lifting lemma.]

B3. Let $p\colon E \to B$ be a covering map, and $p(e_0) = b_0$.

(a) Show that $p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.

(b) Show that the subgroup $p_*(\pi_1(E, e_0))$ in $\pi_1(B, b_0)$ consists of the homotopy classes of loops in B based at b_0 whose liftings in E starting at e_0 are loops.

C Mixed (1 problem)

C1. Suppose that $f\colon X \to Y$ is a continuous bijection.

(a) If X is compact and Y is Hausdorff, show that f is a homeomorphism.

(b) Give an example of topological spaces X and Y and a continuous bijection $f\colon X \to Y$ that is not a homeomorphism.

C2. Let $X = S^1 \vee S^1$ be a wedge of two circles (a figure eight – two circles joined at a single point).

(a) Determine the fundamental group of X using the fact that the fundamental group of S^1 is isomorphic to the integers, $\mathbb{Z}$.

(b) Give the definition of a covering space, and explicitly construct a 3-fold covering space of X.